Global Journal of Research and Review (GJRR)
www.gjrr.org

Original Article

Common Fixed Point Theorem in Sequentially Complete Hausdorff Ordered Uniform Spaces

Yogita R Sharma*

Department of Mathematics, Saffrony Institute of Technology, Linch, Mehsana, India

*Corresponding author e-mail: yogitasharma17@yahoo.com

ABSTRACT

In this paper we obtain coincidence point and common fixed point theorem for contraction type mappings satisfying a contractive inequality using generalized altering distance function in ordered uniform spaces. In this paper I considering sequentially complete Hausdorff ordered Uniform space, four sequentially continuous mappings and their pairs are compatible and two mappings are increasing with respect to other two.

Keywords: Ordered uniform space, Coincident point, Common fixed point, Compatible pair of mappings, Altering distance function, Generalized weakly C-contraction.

INTRODUCTION

The well known Banach fixed point theorem for contraction mapping has been generalized and extended in many directions. Since the uniform spaces form a natural extension of the metric spaces, there exists a considerable literature of fixed point theory dealing with results on fixed or common fixed points in uniform spaces.

A new category of fixed point problems was addressed by Khan *et al*[12].They introduced the notion of an altering distance function which is a control function that alters distance between two points in a metric space.

Definition 1.1[12]

The function $\psi:[0,+\infty)\to[0,+\infty)$ is called an altering distance function, if the following properties are satisfied:

(i) ψ is continuous and non decreasing,

(ii) $\psi(t)=0$ if and only if $t=0$.

Altering distance has been used in metric fixed point theory in recent papers[3,6,11,14]. Choudhury[2] also introduced the following definition.

Definition 1.2[2]

A $f:X\to X$ mapping , where (X,d) is a metric space is said to be weakly C-contractive if for all $x,y\in X$, the following inequality holds:

$$d(fx, fy) \le \frac{1}{2}[d(x, fy) + d(y, fx)] - \varphi(d(x, fy), d(y, fx))$$

Where $\varphi: [0, +\infty)^2 \to [0, +\infty)$ is a continuous function such that $\varphi(s, t) = 0$ if and only if $s = t = 0$.

In[2] the author proves that if X is complete then every weak C- contraction has a unique fixed Point. Also fixed point theorems in partially ordered spaces and sequentially complete Hausdorff ordered uniform spaces are given in[1,4,5,7,9,13].

In this paper we establish some coincidence and common fixed point results for four self mappings on a Hausdorff sequentially complete ordered uniform spaces satisfying a generalized weak C-contractive condition which involves altering distance function.

Now, we recall some relevant definitions and properties.

We call a (X, U) pair to be a uniform space which consists of a non empty set X together with a uniformity U. It is well known (see Dugundji[8] and Kelley[10] that any uniform structure U on X is induced by a family D of pseudometrics on X and conversely any family D of pseudometrics on a set X induces on X a structure of uniform space U. In addition, U is Hausdorff if and only if D is separating. A family $D = \{d_\alpha : \alpha \in A\}$ of pseudometrics on X is said to be separating if for each pair of points $x, y \in X$ with $x \neq y$, there is a $\alpha \in A$ such that $d_\alpha(x, y) \neq 0$.

Consider a uniform space (X, U) with a uniformity U induced by a family $D = \{d_\alpha : \alpha \in A\}$ of pseudometrics on X. A sequence $\{x_n\}$ of elements in X is said to be Cauchy if for every $\varepsilon > 0$ and $\alpha \in A$, there is an N with $d_\alpha(x_n, x_{n+k}) < \varepsilon$ for all $n \ge N$ and $k \in Z^+$. The sequence $\{x_n\}$ is called convergent if there exists an $x_0 \in X$ such for every $\varepsilon > 0$ and $\alpha \in A$, there is an N with $d_\alpha(x_n, x_0) < \varepsilon$ for all $n \ge N$. A uniform space is called sequentially complete if any Cauchy sequence is convergent. A subset of X is said to be sequentially closed if it contains the limit of any convergent sequence of its elements.

Let X be a non-empty set, $f, g, h, s: X \to X$ are given self mappings on X. If $w = fx = gx = hx = sx$ for some $x \in X$, then x is called a coincidence point of f, g, h and s, and w is called a point of coincidence of f, g, h and s. If $w = x$, then x is called a common fixed point of f, g, h and s.

Definition 1.3[7]

Let $(X, \prec)$ be a partially ordered set. Two mappings are said to be weakly increasing if $fx \prec gfx$ and $gx \prec fgx$ for all $x \in X$.

Let X be a non-empty set and $h, s: X \to X$ be a given mapping. For every $x \in X$, we denote by $h^{-1}(x)$ and $s^{-1}(x)$ the subset of X defined by:

$$h^{-1}(x) = \{u \in X | h(u) = x\}$$

And $$s^{-1}(x) = \{u \in X | s(u) = x\}$$

Definition 1.4

Let $f, g, h, s: X \to X$ are given self mappings on X. The pair (f, g) is said to be compatible if $\lim_{n\to\infty} d_\alpha(fgx_n, gfx_n) = 0$ for each $\alpha \in A$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

MAIN RESULT

Theorem 2.1

Let $(X, U, \prec)$ be a sequentially complete Hausdorff ordered uniform space. Let $f, g, h, s: X \to X$ be given mapping satisfying.

(i) $fX \subseteq hX, gX \subseteq kX$,

(ii) f, g, h and s are sequentially continuous,

(iii) the pairs (f, h) and (g, k) are compatible,

(iv) f and g are weakly increasing with respect to h and k.

Suppose that for every $\alpha \in A$ and $x, y \in X$ such that hx and ky are comparable, we have.

$$\psi_\alpha(d_\alpha(fx, gy)) \le \psi_\alpha\left(\frac{1}{2}[d_\alpha(hx, gy) + d_\alpha(ky, fx)]\right) - \varphi_\alpha(d_\alpha(hx, gy) + d_\alpha(ky, fx)) \quad \ldots 1$$

Where for each $\alpha \in A, \psi_\alpha: [0, +\infty) \to [0, +\infty)$ is an altering distance function and $\varphi_\alpha: [0, +\infty)^2 \to [0, +\infty)$ is a continuous function with $\varphi_\alpha(s, t) = 0$ if and only if $t = s = 0$.

Then f, g, h and k have a coincidence point $u \in X$, that is, $fu = gu = hu = ku$.

Proof

Let x_0 be an arbitrary point in X. Since $fX \subseteq hX$, there exists $x_1 \in X$ such that $hx_1 = fx_0$. Since $gX \subseteq kX$, there exists $x_2 \in X$ such that $kx_2 = gx_1$. Continuing this process, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X defined by.

$$y_{2n} = fx_{2n} = hx_{2n+1}, \quad y_{2n+1} = gx_{2n+1} = kx_{2n+2} \quad \forall n \in Z^+ \quad \ldots 2$$

By construction, we have $x_{2n+1} \in h^{-1}(fx_{2n})$ and $x_{2n+2} \in k^{-1}(gx_{2n+1})$, then using the fact that f and g are weakly increasing with respect to h and k, we obtain.

$$hx_{2n+1} = fx_{2n} \prec gx_{2n+1} = kx_{2n+2} \quad \forall n \in Z^+ \cup \{0\}$$

$$kx_{2n+2} = gx_{2n+1} \prec fx_{2n+2} = hx_{2n+3}$$ Then

$$hx_1 \prec hx_2 \prec \cdots \prec hx_n \prec hx_{n+1} \prec \cdots$$ Or

$$y_0 \prec y_1 \prec \cdots \prec y_{n-1} \prec y_n \prec \cdots$$

Since hx_{2n} and hx_{2n+1} are comparable for each $\alpha \in A$ by inequality (1), we have.

$$\psi_\alpha(d_\alpha(y_{2n}, y_{2n+1})) = \psi_\alpha(d_\alpha(fx_{2n}, gx_{2n+1}))$$

$$\le \psi_\alpha\left(\frac{1}{2}[d_\alpha(hx_{2n}, gx_{2n+1}) + d_\alpha(fx_{2n}, kx_{2n+1})]\right)$$

$$-\varphi_\alpha(d_\alpha(hx_{2n}, gx_{2n+1}), \quad d_\alpha(fx_{2n}, kx_{2n+1}))$$

$$= \psi_\alpha\left(\frac{1}{2}[d_\alpha(y_{2n-1}, y_{2n+1}) + d_\alpha(y_{2n}, y_{2n})]\right) \quad \ldots 3$$

$$-\varphi_\alpha(d_\alpha(y_{2n-1}, y_{2n+1}), d_\alpha(y_{2n}, y_{2n}))$$

$$= \psi_\alpha\left(\frac{1}{2}d_\alpha(y_{2n-1}, y_{2n+1})\right) - \varphi_\alpha(d_\alpha(y_{2n-1}, y_{2n+1}), 0)$$

$$\le \psi_\alpha\left(\frac{1}{2}d_\alpha(y_{2n-1}, y_{2n+1})\right)$$

Since ψ_α is a non decreasing function, we get that.

$$d_\alpha(y_{2n}, y_{2n+1}) \le \frac{1}{2}d_\alpha(y_{2n-1}, y_{2n+1})$$

By triangular inequality, we have.

$$d_\alpha(y_{2n-1}, y_{2n+1}) \le d_\alpha(y_{2n-1}, y_{2n}) + d_\alpha(y_{2n}, y_{2n+1}) \quad \ldots 4$$

Thus

$$d_\alpha(y_{2n}, y_{2n+1}) \le d_\alpha(y_{2n-1}, y_{2n}) \quad \ldots 5$$

$$d_\alpha(y_{2n+1}, y_{2n+2}) \le d_\alpha(y_{2n}, y_{2n+1}) \quad \ldots 6$$

$$d_\alpha(y_{n+1}, y_{n+2}) \le d_\alpha(y_n, y_{n+1}) \quad \ldots 7$$

It follows that the sequence $\{d_\alpha(y_n, y_{n+1})\}$ is monotonic decreasing. Hence, there exists $r \ge 0$ such that.

$$\lim_{n\to\infty} d_\alpha(y_n, y_{n+1}) = r \quad \ldots 8$$

By (4) we have

$$d_\alpha(y_{2n}, y_{2n+1}) \le \frac{1}{2}d_\alpha(y_{2n-1}, y_{2n+1}) \le \frac{1}{2}(d_\alpha(y_{2n-1}, y_{2n}) + d_\alpha(y_{2n}, y_{2n+1})) \quad \ldots 9$$

Taking $n \to \infty$ and using (8), we get-

$$r \le \lim_{n\to\infty} \frac{1}{2}d_\alpha(y_{2n-1}, y_{2n+1}) \le \frac{1}{2}(r + r) \quad \ldots 10$$

$$\lim_{n\to\infty} \frac{1}{2}d_\alpha(y_{2n-1}, y_{2n+1}) = 2r \quad \ldots 11$$

Taking $n \to \infty$ in (3) and using the continuity of $\psi_\alpha, \varphi_\alpha$ and (8), (11) we get that-

$$\psi_\alpha(r) \le \psi_\alpha\left(\frac{1}{2}(2r)\right) - \varphi_\alpha(2r, 0) \le \psi_\alpha(r)$$

Which implies that $\varphi_\alpha(2r, 0) = 0$ and hence $r = 0$, so we have.

$$\lim_{n\to\infty} d_\alpha(y_n, y_{n+1}) = 0 \quad \forall \alpha \in A \quad \ldots 12$$

To prove that $\{y_n\}$ is a Cauchy sequence in X, it is sufficient to show that $\{y_{2n}\}$ is a Cauchy sequence. Suppose to the contrary, that $\{y_n\}$ is not a Cauchy sequence. Then there exists $\alpha_0 \in A$ and $\varepsilon > 0$ for which we can find two subsequences $\{y_{2m(k)}\}$ and $\{y_{2n(k)}\}$ of $\{y_n\}$ such that $n(k)$ is the smallest index for which,

$$n(k) > m(k) > k,\ d_{\alpha_0}(y_{2m(k)}, y_{2n(k)}) \geq \varepsilon \quad \text{..........13}$$

This means that,

$$d_{\alpha_0}(y_{2m(k)}, y_{2n(k)-2}) < \varepsilon \quad \text{...........14}$$

Therefore, we use (13), (14) and triangular inequality to get,

$$\varepsilon \leq d_{\alpha_0}(y_{2m(k)}, y_{2n(k)})$$
$$\leq d_{\alpha_0}(y_{2m(k)}, y_{2n(k)-2}) + d_{\alpha_0}(y_{2n(k)-2}, y_{2n(k)-1}) + d_{\alpha_0}(y_{2n(k)-1}, y_{2n(k)})$$
$$\leq \varepsilon + d_{\alpha_0}(y_{2n(k)-2}, y_{2n(k)-1}) + d_{\alpha_0}(y_{2n(k)-1}, y_{2n(k)})$$

Taking $k \to \infty$ in the above inequality and using (12), we find,

$$\lim_{n\to\infty} d_{\alpha_0}(y_{2m(k)}, y_{2n(k)}) = \varepsilon \quad \text{......................15}$$

On the other hand, we have,

$$|d_{\alpha_0}(y_{2m(k)-1}, y_{2n(k)}) - d_{\alpha_0}(y_{2m(k)}, y_{2n(k)})| \leq d_{\alpha_0}(y_{2m(k)-1}, y_{2n(k)})$$

Taking $k \to \infty$ in the above inequality and using (12), (15) we find,

$$\lim_{n\to\infty} d_{\alpha_0}(y_{2m(k)-1}, y_{2n(k)}) = \varepsilon \quad \text{................16}$$

On the other hand, we have,

$$d_{\alpha_0}(y_{2m(k)}, y_{2n(k)}) \leq d_{\alpha_0}(y_{2m(k)}, y_{2n(k)+1}) + d_{\alpha_0}(y_{2n(k)+1}, y_{2n(k)})$$

Taking $k \to \infty$ in the above inequality and using (12), (15) we have,

$$\varepsilon \leq \lim_{n\to\infty} d_{\alpha_0}(y_{2m(k)}, y_{2n(k)+1}) \quad \text{................17}$$

Also, by triangular inequality, we have,

$$d_{\alpha_0}(y_{2m(k)}, y_{2n(k)}) \leq d_{\alpha_0}(y_{2m(k)}, y_{2m(k)-1}) +$$
$$d_{\alpha_0}(y_{2m(k)-1}, y_{2n(k)+1}) + d_{\alpha_0}(y_{2n(k)+1}, y_{2n(k)})$$

Taking again $k \to \infty$ in the above inequality and using (12), (15) and (16) we find,

$$\varepsilon \leq \lim_{n\to\infty} d_{\alpha_0}(y_{2m(k)-1}, y_{2n(k)+1})$$

Similarly, we can show that,

$$\lim_{n\to\infty} d_{\alpha_0}(y_{2m(k)-1}, y_{2n(k)+1}) \leq \varepsilon$$
$$\lim_{n\to\infty} d_{\alpha_0}(y_{2m(k)-1}, y_{2n(k)+1}) = \varepsilon \quad \text{..............18}$$

From (1), we have,

$$\psi_{\alpha_0}(d_{\alpha_0}(y_{2n(k)+1}, y_{2m(k)})) = \psi_{\alpha_0}(d_{\alpha_0}(fx_{2n(k)}, gx_{2m(k)-1}))$$
$$\leq \psi_{\alpha_0}\left(\frac{1}{2}[d_{\alpha_0}(y_{2n(k)}, gx_{2m(k)-1}) + d_{\alpha_0}(fx_{2n(k)}, y_{2m(k)-1})]\right)$$
$$-\varphi_{\alpha_0}(d_{\alpha_0}(y_{2n(k)}, gx_{2m(k)-1}), d_{\alpha_0}(fx_{2n(k)}, y_{2m(k)-1})) \quad \text{..19}$$
$$= \psi_{\alpha_0}\left(\frac{1}{2}[d_{\alpha_0}(y_{2n(k)}, y_{2m(k)}) + d_{\alpha_0}(y_{2n(k)+1}, y_{2m(k)-1})]\right)$$
$$-\varphi_{\alpha_0}(d_{\alpha_0}(y_{2n(k)}, y_{2m(k)}), d_{\alpha_0}(y_{2n(k)+1}, y_{2m(k)-1}))$$
$$\leq \psi_{\alpha_0}\left(\frac{1}{2}[d_{\alpha_0}(y_{2n(k)}, y_{2m(k)}) + d_{\alpha_0}(y_{2n(k)+1}, y_{2m(k)-1})]\right)$$

Since ψ_{α_0} is a non decreasing function, we get that,

$$\psi_{\alpha_0}(d_{\alpha_0}(y_{2n(k)+1}, y_{2m(k)})) \leq \frac{1}{2}[d_{\alpha_0}(y_{2n(k)}, y_{2m(k)}) + d_{\alpha_0}(y_{2n(k)+1}, y_{2m(k)-1})]$$

Taking again $k \to \infty$ in the above inequality and using (15), (18) we find,

$$\lim_{n\to\infty} d_{\alpha_0}(y_{2n(k)+1}, y_{2m(k)}) \leq \varepsilon \quad \text{................20}$$

Therefore, from (17) and (20) we have,

$$\lim_{n\to\infty} d_{\alpha_0}(y_{2n(k)+1}, y_{2m(k)}) = \varepsilon \quad \text{................21}$$

Taking $k \to \infty$ in (19) and using (15), (18), (21) and the continuity of we find ψ_{α_0} and φ_{α_0}, we get that,

$$\psi_{\alpha_0}(\varepsilon) \leq \psi_{\alpha_0}\left(\frac{1}{2}(\varepsilon + \varepsilon)\right) - \varphi_{\alpha_0}(\varepsilon, \varepsilon) \leq \psi_{\alpha_0}\left(\frac{1}{2}(\varepsilon + \varepsilon)\right)$$

Which implies that $\varphi_{\alpha_0}(\varepsilon, \varepsilon)$ and hence $= 0$, a contradiction. Thus $\{y_{2n}\}$ is a Cauchy sequence and hence $\{y_n\}$ is a Cauchy sequence. Since (X, U) sequentially complete Hausdorff uniform space, there is $u \in X$ such that,

$$\lim_{n\to\infty} y_n = u$$

Therefore,

$$d_\alpha(y_n, u) \to 0 \qquad \forall \alpha \in A \quad \text{....................22}$$

From the sequentially continuity of h and k, we get,

$$\lim_{n\to\infty} h(y_n) = hu$$

Therefore,

$$d_\alpha(h(y_n), hu) \to 0. \qquad \forall \alpha \in A \quad \text{.....................23}$$

The triangular inequality and (2) yields,

$d_\alpha(hu,fu) \le d_\alpha(hu,h(hx_{2n+1})) + d_\alpha(h(fx_{2n}),f(hx_{2n})) + d_\alpha(f(hx_{2n}),fu)$
$d_\alpha(ku,gu) \le d_\alpha(ku,h(kx_{2n+2})) + d_\alpha(k(gx_{2n}),g(kx_{2n+1})) + d_\alpha(g(kx_{2n+1}),gu)$
$\forall \alpha \in A$ …………………………………..….24

From (2) and (22),
$d_\alpha(hx_{2n},u) \to 0,\quad d_\alpha(fx_{2n},u) \to 0,\quad \forall \alpha \in A$
$d_\alpha(kx_{2n+1},u) \to 0,\quad d_\alpha(gx_{2n+1},u) \to 0,$ ………………25

The pair (f,h) and (g,k) is compatible, then,
$d_\alpha(h(fx_{2n}),f(hx_{2n})) \to 0,\quad d_\alpha(k(gx_{2n+1}),g(kx_{2n+1})) \to 0.\quad \forall \alpha \in A$
……………………………………………26

Using the sequentially continuity of f,g and (25), we have,
$d_\alpha(f(hx_{2n}),fu) \to 0,\quad d_\alpha(g(kx_{2n+1}),gu) \to 0.\quad \forall \alpha \in A$
……………………………………………27

Combining (23), (26) together with (27) and taking $n \to \infty$ in (24), we obtain,
$d_\alpha(hu,fu) \le 0,\quad d_\alpha(ku,gu) \le 0 \ \forall \alpha \in A$

Which means that $hu = fu$ and $gu = ku$. So u is a coincidence point of f,g,h and k.

REFERENCES

1. A. Amini-Harandi and H. Emami, 2010. A fixed point theorem for contraction type maps in partially ordered metric spaces application to ordinary differential equations. *Nonlinear Anal*., 72: 2238-2242.
2. B.S. Choudhury, 2009. Unique fixed point theorem for weak C- contractive mappings. Kathmandu Univ. *J. Sci. Eng. Tech*., 5(1): 6-13.
3. G.V.R. Babu, B. Lalitha and M.L. Sandhya, 2007. Common fixed point theorems involving two generalized altering distance functions in four variables. *Proc. Jangeon Math. Soc*., 10: 83-93.
4. H. Ayadi, 2011. Coincidence and common fixed point result for contraction type maps in partially ordered metric spaces. *Int. J. Math. Anal*., 5(13): 631-642.
5. H. Ayadi, H.K. Nashine, B. Samet and H. Yazidi, 2011. Coincidence and common fixed point results in partially ordered cone metric spaces and applications to integral equations. *Nonlinear Anal*., 74: 6814-6825.
6. H.K. Nashine, B. Samet, 2011. Fixed point results for mappings satisfying (ψ,φ)-weakly contractive condition in partially ordered metric spaces. *Nonlinear Anal*., 74: 2201-2209.
7. I. Altun and H. Simsek, 2010. Some fixed point theorems on ordered metric spaces and applications. Fixed point theory and applications, 2010 Article ID 621469: 17 pages.
8. J. Dugundji, 1966. *Topology*. Allyn and Bacon, Massachusetts USA.
9. J. Esmaily, S.M. Vaezpour, 2012. Coincidence and common fixed point results for generalized weakly C-contraction in ordered uniform spaces. *J. Basic. Appl. Sci. Res*., 2(4): 4139-4148.
10. J.L. Kelly, 1955. General topology. Van Nostrand Princeton, New York.
11. K.P.R. Sastry and G.V.R. Babu, 1999. Some fixed point theorems by altering distances between the points. *Indian J. Pure Appl. Math*., 30: 641-647.
12. M.S. Khan, M. Swaleh and S. Sessa, 1984. Fixed point theorems by altering distances between the points. *Bull. Austral. Math. Soc*., 30(1): 1-9.
13. R.P. Agrawal, M.A. El-Gebeily and D. O'Regan, 2008. Generalized contractions in partially ordered metric spaces. *Appl. Anal*., 87:109-116.
14. S.V.R. Naidu, 2003. Some fixed point theorems in metric spaces by altering distances, *Czechoslovak Math. J*., 53: 205-212.