Surface waves in non-homogeneous fibre-reinforced anisotropic elastic media with voids

Aftab Khan, A. I. Anya and Hajra Kaneez

Department of Mathematics, COMSATS, Institute of Information Technology, Park Road, Chak Shahzad, Islamabad, Pakistan

ABSTRACT

This paper investigates the propagation of surface waves in a non homogeneous fibre-reinforced elastic media with voids. The general surface wave speed is derived to study the effects of voids on surface waves in non homogeneous fibre-reinforced elastic solid and discussed its particular cases for Stoneley, Love and Rayleigh waves. The results obtained in this investigation are more general in the sense that some earlier published results are obtained from our result as special cases. Also by neglecting non homogeneity and the reinforced elastic parameters, the results reduce to well known isotropic medium.

Key words: Surface waves, non-homogeneous, fiber reinforced, anisotropic, voids.

INTRODUCTION

There are many types of surface waves [1-6] but we only focus on Stoneley, Love and Rayleigh waves. In earthquake the movement is due to the surface waves. These are also used for detecting cracks and other defects in materials. Lord Rayleigh [3] was the first to observe such kind of waves in 1885. That’s why we called it Rayleigh waves. Sengupta and Nath [7] investigated surface waves in fibre-reinforced anisotropic elastic media but their decomposition of displacement vector was not correct that’s why some errors are found in their investigations [8].

The idea of continuous self-reinforcement at every point of an elastic solid was introduced by Belfield [9]. The superiority of fibre-reinforced composite materials over other structural materials attracted many authors to study different type of problems in this field. Fibre-reinforced composite structures are used due to their low weight and high strength. Two important components namely concrete and steel of a reinforced medium are bound together as a single unit so that there can be no relative displacement between them i.e. they act together as a single anisotropic unit. The artificial structures on the surface of the earth are excited during an earthquake, which give rise to violent vibrations in some cases [10, 11]. Engineers and architects are in search of such reinforced elastic materials for the structures that resist the oscillatory vibration. The propagation of waves depends upon the ground vibration and the physical properties of the structure material. Kakar et al. [12-16] discussed surface wave propagation in non homogeneous media.

In classical theory of elasticity, the voids is an important generalization. Nunziato and Cowin [17] and Cowin and Nunziato [18] discusse the theory in elastic media with voids. Puri and Cowin [19] studied the effects of voids on plane waves in linear elastic media and it is evident that pure shear waves remain unaffected by the presence of pores. Chandrasekhariah [20] and [21] discussed the effects of voids on propagation of surface and plane waves respectively.

Aim of this paper is to investigate the propagation of surface waves in a non homogeneous fibre-reinforced elastic media with voids. The general surface wave speed is derived to study the effect of voids on surface waves. Particular
cases for Stonely, Love and Rayleigh waves are discussed. The results obtained in this investigation are more general in the sense that some earlier published results are obtained from our result as special cases. For homogeneous medium our results are well agreement to fibre-reinforced materials. It is also observed that the corresponding classical results follow from this analysis, in homogeneous media, by neglecting reinforced parameters. Results for homogeneous media can be deduced from this investigation.

2. Formulation of the Problem:
Medium is consisting of two non-homogeneous anisotropic fibre-reinforced semi-infinite elastic solid media $M_1$ and $M_2$ with different elastic and reinforcement parameters. The non-homogeneity of the material is depending on the space variable. It is assumed that non-homogeneity grows or decays slowly. Its rate of growth or decay is proportional to its value at that point i.e.

$$\frac{d\lambda}{dx} \propto \lambda; \quad \text{where } \lambda \text{ is an elastic parameter. This implies}$$

$$\frac{d\lambda}{dx} = m\lambda,$$

where $m$ is a constant, which is positive for non-homogeneity growth and negative for decay.

Above equation implies

$$\lambda = \lambda_0 e^{mx_2}$$

For $m = 0$, $\lambda = \lambda_0$. Thus for $m = 0$, the medium is homogeneous.

The two media are perfectly welded in contact at a plane interface. Let us take orthogonal Cartesian axes $\alpha x_1 x_2 x_3$ with the origin at $\alpha x_1$. $\alpha x_1$ is pointing vertically upwards into the medium $M_1 (x_2 \geq 0)$. Each of the media $M_1 (x_2 \geq 0)$ and $M_2 (x_2 \leq 0)$ separated at $x_2 = 0$. Both media are rotating about an axis.

It is assumed that the waves travel in the positive direction of the $x_1$-axis and at any instant, all particles have equal displacements in any direction parallel to $\alpha x_3$. In view of that assumptions, the propagation of waves will be independent of $x_3$.

In the presence of voids, the general equation for a fibre-reinforced linearly elastic anisotropic media w.r.t. a direction $\tilde{a}_1, \tilde{a}_2, \tilde{a}_3$ is as under

$$\sum_{ijkl} \tau_{ij} \delta_{ij} + 2\mu_l \epsilon_{ij} + \alpha(a_k a_m \epsilon_{km} \delta_{ij} + \epsilon_{ik} a_j a_l) + 2(\mu_L - \mu_T)(a_k a_l \epsilon_{ij} + a_j a_k \epsilon_{lj})$$

$$\beta(a_k a_m \epsilon_{km} a_i a_j) + \beta \delta_{ij} \phi,$$

Where, $\phi$ are components of stress and strain tensor is $\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ - and $\lambda, \mu_T$ are elastic parameters.

$\tilde{a}_i \phi$ and $\tilde{a}_i \mu_L \phi \tilde{a}_i$ are reinforced anisotropic elastic parameters, $u_i$ are the displacement vectors components.

In the absence of body forces, the field equations may be taken as follows:

$$\tau_{ij} = \lambda \epsilon_{ik} \delta_{ij} + 2\mu_l \epsilon_{ij} + \alpha(a_k a_m \epsilon_{km} \delta_{ij} + \epsilon_{ik} a_j a_l) + 2(\mu_L - \mu_T)(a_k a_l \epsilon_{ij} + a_j a_k \epsilon_{lj})$$

$$\beta(a_k a_m \epsilon_{km} a_i a_j) + \lambda \delta_{ij} \phi,$$

$$\tau_{ij,j} = \rho u_i$$

$$\alpha \phi_{ij} - \alpha_t \phi - \alpha \phi_j - \xi u_{i,j} = \rho \ddot{\phi}.$$

In these equations, $\phi$ is the so-called volume fraction field. $\alpha, \beta, \omega, \sigma$ and $\kappa$ are new material constants characterizing the presence of voids. Where $\epsilon_{ijk}$ is the Levi-Civita tensor, $\tau_{ij}$ are components of stress, $\rho$ is the mass density and $u_i$ is the displacement vector. Comma followed by index shows partial derivative with respect to coordinate. Also Einstein summation convention over repeated indexes is used.

The propagation equations of small elastic disturbances are as follows.

In component form, the equation of motion in the presence of voids becomes
\[ \begin{align*}
\tau_{11,1} + \tau_{12,2} + \xi \phi_1 &= \rho \dddot{u}_1, \\
\tau_{21,1} + \tau_{22,2} + \xi \phi_2 &= \rho \dddot{u}_2, \\
\tau_{31,1} + \tau_{32,2} &= \rho \dddot{u}_3, \\
\alpha (\phi_{11} + \phi_{22}) - \omega_0 \phi - \omega \phi - \xi (u_{11} + u_{22}) &= \rho \kappa \dddot{\phi} 
\end{align*} \tag{2.1} \]

In the present problem we consider exponentially decaying non-homogeneous material. Hence density, elastic module and elastic parameters may be taken in the following form.

\[ \begin{align*}
\rho &= \rho_0 e^{-mx_2} \\
\lambda &= \lambda_0 e^{-mx_2} \\
\mu &= \mu_0 e^{-mx_2} \\
\alpha &= \alpha_0 e^{-mx_2} \\
\beta &= \beta_0 e^{-mx_2} 
\end{align*} \]

We choose the fibre direction as \( \vec{a} = (1, 0, 0) \).

Now using the above equations and taking all derivatives w.r.t. \( x_1 \) zero. The equation (2.1a) becomes

\[ \begin{align*}
(\lambda + 2\alpha + 4\mu_1 - 2\mu_T + \beta)u_{1,11} + (\alpha + \lambda + \mu_L)u_{2,21} + \mu_L u_{1,12} - m\mu_T (u_{1,12} + u_{2,1}) + \xi \phi_1 &= \rho \dddot{u}_1, 
\end{align*} \tag{2.2a} \]

similarly equations (2.1b) and (2.1c) takes the following form

\[ \begin{align*}
(\alpha + \lambda + \mu_L)u_{1,12} + \mu_L u_{2,12} + m(\lambda + \alpha)u_{1,1} - m(\lambda + 2\mu_T)u_{1,2} + \xi \phi_2 &= \rho \dddot{u}_2, \\
\mu_L u_{3,11} + \mu_L u_{3,22} - m\mu_T u_{3,2} &= \rho \dddot{u}_3, \\
\alpha (\phi_{11} + \phi_{22}) - \omega_0 \phi - \omega \phi - \xi (u_{11} + u_{22}) &= \rho \kappa \dddot{\phi}. 
\end{align*} \tag{2.2b, 2.2c} \]

Similarly, we can get similar relations in \( \mathbf{M}_1 \) with \( \xi \alpha, \lambda, \Phi \) and \( \xi \) are replaced by \( \rho', \alpha', \lambda', \mu'_T \) and \( \beta' \).

### 3. Solution of the Problem

We seek harmonic solutions in the form

\[ u_{1,1}, u_{2,2}, u_{3,2} = \hat{u}_1 e^{i\omega_1 x_1}, \hat{u}_2 e^{i\omega_2 x_1}, \hat{u}_3 e^{i\omega_3 x_1} \exp[i \omega T x_2] \]

Thus equations (2.2a, b) of motion becomes

\[ \begin{align*}
\left[ \mu_L D^2 - m\mu_T D + \omega^2 \left\{ \rho c^2 - (\lambda + 2\alpha + 4\mu_L - 2\mu_T + \beta) \right\} \right] \hat{u}_1 + i\omega [\alpha + \lambda + \mu_L] D - m\mu_T \right] \hat{u}_1 + i\omega \xi \hat{\phi} = 0 \\
\left[ (\lambda + 2\mu_T) D^2 - m(\lambda + 2\mu_T) D + \omega^2 \left\{ \rho c^2 - \mu_L \right\} \right] \hat{u}_2 + i\omega [\alpha + \lambda + \mu_L] D - m(\alpha + \lambda) \right] \hat{u}_1 + \xi D \hat{\phi} = 0. 
\end{align*} \tag{3.1a, 3.1b} \]

\[ \begin{align*}
\mu_L u_{3,11} + \mu_L u_{3,22} - m\mu_T u_{3,2} &= \rho \dddot{u}_3, \\
\left\{ \alpha (D^2 - \omega^2) - \omega_0 + i\alpha \omega \sigma + \omega^2 \rho c^2 \right\} \hat{\phi} - \xi (i\alpha \ddot{u}_1 + D \hat{u}_2) &= 0. 
\end{align*} \tag{3.1c, 3.1d} \]

Where, \( D = \frac{d}{dx_2} \)

Similarly we can get similar relations in \( \mathbf{M}_1 \) with \( \xi \alpha, \lambda, \Phi \) and \( \xi \) are replaced by \( \rho', \alpha', \lambda', \mu'_T \) and \( \beta' \).

Thus coupled equations (3.1a,b,d) becomes

\[ \begin{align*}
(h_1 D^2 - m\lambda \right) D - \omega^2 h_3 + \omega^2 \rho c^2 \right) \dddot{u}_1 + i\omega (h_2 D - m\lambda) \dddot{u}_2 + i\omega \xi \dddot{\phi} = 0, 
\end{align*} \]
\[(h_4 D^2 - m h_4 D - \omega^2 \eta_1 + \omega^2 \rho c^2) \hat{u}_1 + i \omega (h_2 D - m (h_2 - \eta_1)) \hat{u}_1 + \xi D \hat{\phi} = 0,\]
\[
\{\alpha (D^2 - \omega^2) - \omega_0 + i \alpha \sigma + \omega^2 c^2 \rho \kappa\} \hat{\phi} - \xi (i \omega \hat{u}_1 + D \hat{u}_2) = 0,
\]
and uncoupled equation (3.1c) becomes
\[
\{h_3 D^2 - m h_3 D - \omega \hat{\sigma} (h_1 - \rho c^2)\} \hat{u}_3 = 0
\]

where
\[
h_1 = \mu_L, \quad h_2 = (\alpha + \lambda + \mu),
\]
\[
h_3 = (\lambda + 2 \alpha + 4 \mu - 2 \mu + \beta),
\]
\[
h_4 = (\lambda + 2 \mu) \text{ and } \quad h_5 = \mu_T.
\]

The uncoupled equation has the following solution,
\[
u_u = \left(E e^{-\eta_1 x} + E_1 e^{-\eta_2 x}\right) e^{i \omega (x - ct)},
\]

where \(\eta_1\) and \(\eta_2\) are roots of the equation \(h_3 \eta^2 - m h_3 \eta - \omega \hat{\sigma} (h_1 - \rho c^2) = 0\).

For positive real root \(\eta_1\), it is necessary that \(0 < 4 \rho c^2 < h_3 m^2 + 4 h_1\) and in the homogeneous medium \(0 < \rho c^2 < h_1\) otherwise transverse component does not exist. For boundedness \(u_3 = E e^{-\eta_1 x} \exp\{i \omega (x - ct)\}\),

Above set of coupled equations can be written as
\[
\begin{pmatrix}
(h_1 D^2 - m h_1 D - A_1) \hat{u}_1 + i \omega (h_2 D - m h_1) \hat{u}_2 + i \omega \xi \hat{\phi} = 0 \\
(h_4 D^2 - m h_4 D - A_2) \hat{u}_2 + i \omega (h_2 D - m (h_2 - h_1)) \hat{u}_1 + \xi D \hat{\phi} = 0 \\
(D^2 - A_3) \hat{\phi} - \xi (i \omega \hat{u}_1 + D \hat{u}_2) = 0
\end{pmatrix}
\]

where
\[
A_1 = \omega^2 \eta_1 - \omega^2 \rho c^2
\]
\[
A_2 = \omega^2 \eta_2 - \omega^2 \rho c^2
\]
\[
A_3 = \omega^2 + \frac{\omega_0 - i \omega \sigma - \omega^2 c^2 \rho \kappa}{\alpha}
\]

From above set of equations, we have
\[
\begin{vmatrix}
(h_1 D^2 - m h_1 D - A_1) & i \omega (h_2 D - m h_1) & i \omega \xi \\
i \omega (h_2 D - m (h_2 - h_1)) & (h_4 D^2 - m h_4 D - A_2) & \xi D \\
-i \omega \xi & -\xi D & (D^2 - A_3)
\end{vmatrix}
(\hat{u}_1, \hat{u}_2, \hat{\phi}) = 0
\]

This implies
\[
(D^6 + C_1 D^5 - C_2 D^4 - C_3 D^3 + C_4 D^2 + C_5 D - C_6)(\hat{u}_1, \hat{u}_2, \hat{\phi}) = 0
\]

where
\[
C_1 = \frac{m h_4}{h_1 h_4} (h_2 - h_1)
\]

\[241\]
\[
C_2 = \frac{1}{h_4 h_4} \left( h_4 A_1 + h_1 (A_2 + h_4 A_3 - \xi^2) - \omega^2 h_2^2 - m h_4 h_5 \right)
\]

\[
C_3 = \frac{m}{h_1 h_4} \left\{ h_2 \left( \xi^2 - A_2 - h_4 A_3 \right) - h_4 A_1 - h_2 (h_1 - h_2) + \omega^2 h_2^2 - h_1 h_4 A_3 \right\}
\]

\[
C_4 = \frac{1}{h_1 h_4} \left\{ \left( A_1 A_2 + h_4 A_1 A_3 + A_2 A_3 - h_2 \omega^2 A_1 \right) - \xi^2 (A_1 + 2 \omega^2 h_2 + h_4 \omega^2) \right\}
\]

\[
C_5 = \frac{m}{h_1 h_4} (h_1 + h_2) h_4
\]

\[
C_6 = \frac{1}{h_1 h_4} \left( A_1 A_2 A_3 - \omega^2 A_2 \xi^2 - m^2 h_5 (h_2 - h_1) \right).
\]

For homogeneous medium, \( m = 0 \), this implies \( C_1 = C_3 = C_5 = 0 \) and \( C_2, C_4 \) and \( C_6 \) must be positive for real positive roots. If there are also no voids then the above equation is easy to solve.

Let \( \alpha_i, i=1,2,..6 \) be six positive real roots, then solution by normal mode method has the following form

\[
\hat{u}_1 = \sum_{n=1}^{6} M_n e^{-\alpha_n x}, \quad \text{(3.3a)}
\]

\[
\hat{u}_2 = \sum_{n=1}^{6} M_{1n} e^{-\alpha_n x}, \quad \text{(3.3b)}
\]

\[
\phi = \sum_{n=1}^{6} M_{2n} e^{-\alpha_n x}, \quad \text{(3.3c)}
\]

where \( M_n, M_{1n} \) and \( M_{2n} \) are some parameters depending on \( c \) and \( \omega \). By using Eqs. (3.2a-c) into Eqs. (3.2), we get the following relations,

\[
M_{1n} = H_{1n} M_n
\]

\[
M_{2n} = H_{2n} M_n
\]

where

\[
H_{1n} = \frac{i \omega \left( A_2 + (h_2 - \h_4) \alpha_n^2 \right) + m(h_2 - \alpha_n h_4)}{h_1 \alpha_n^2 + (h_2 \alpha^2 - A_4) \alpha_n + m h_2 \alpha_n^2 + m \omega(h_2 - h_1)},
\]

\[
H_{2n} = \frac{\alpha_n^2 - A_3}{\xi (\alpha_n H_{1n} - i \omega)} \quad \text{for} \quad n = 1, 2, 3, 4, 5, 6.
\]

Hence we obtain the expressions of the displacement components function and stresses as follows

\[
u_1 = \sum_{n=1}^{6} M_n e^{-\alpha_n x} \exp \{i \omega(x_1 - ct)\},
\]

\[
u_2 = \sum_{n=1}^{6} H_{1n} M_n e^{-\alpha_n x} \exp \{i \omega(x_1 - ct)\},
\]

\[
u_3 = E e^{-\alpha_n x} \exp \{i \omega(x_1 - ct)\},
\]

\[
\phi = \sum_{n=1}^{6} H_{2n} M_n e^{-\alpha_n x} \exp \{i \omega(x_1 - ct)\}.
\]

Also it is found that

\[
\tau_{12} = \sum_{n=1}^{6} h_1 (-\alpha_n + i \omega H_{1n}) M_n e^{-\alpha_n x} \exp \{i \omega(x_1 - ct)\}
\]
\[
\tau_{22} = \sum_{n=1}^{6} \left[ i\omega (h_{2} - h_{1}) - h_{4}\alpha_{n} H_{1n} + \xi H_{2n} \right] M_{n} e^{-\alpha_{n} x_{2}} \exp \left\{ i\omega (x_{1} - ct) \right\}
\]

\[
\tau_{23} = -\eta \omega E h_{3} e^{-\eta x_{2}} \exp \left\{ i\omega (x_{1} - ct) \right\}.
\]

Similar expressions can be obtained for second medium and present them with dashes as follows

\[
u_{1}' = \sum_{n=1}^{6} M_{n}' e^{-\alpha_{n} x_{2}} \exp \left\{ i\omega (x_{1} - ct) \right\},
\]

\[
u_{2}' = \sum_{n=1}^{6} H_{1n}' M_{n}' e^{-\alpha_{n} x_{2}} \exp \left\{ i\omega (x_{1} - ct) \right\},
\]

\[
u_{3}' = Fe^{-\eta x_{2}} \exp \left\{ i\omega (x_{1} - ct) \right\},
\]

\[
\phi_{1}' = \sum_{n=1}^{6} H_{2n}' M_{n}' e^{-\alpha_{n} x_{2}} \exp \left\{ i\omega (x_{1} - ct) \right\}.
\]

Also it is found that

\[
\tau_{12}' = \sum_{n=1}^{6} h_{1}' (-\alpha_{n}' + i\omega H_{1n}') M_{n}' e^{-\alpha_{n} x_{2}} \exp \left\{ i\omega (x_{1} - ct) \right\},
\]

\[
\tau_{22}' = \sum_{n=1}^{6} \left[ i\omega (h_{2} - h_{1}') - h_{4}'\alpha_{n} H_{1n}' + \xi'H_{2n}' \right] M_{n}' e^{-\alpha_{n} x_{2}} \exp \left\{ i\omega (x_{1} - ct) \right\},
\]

\[
\tau_{23}' = -\eta' \omega F h_{3} e^{-\eta x_{2}} \exp \left\{ i\omega (x_{1} - ct) \right\}.
\]

In order to determine the secular equations, we have the following boundary conditions.

**4. Boundary conditions**

I. The displacement components and their rate of change w.r.t. \( x_{2} \), between the mediums are continuous, i.e.

\[
u_{1} = u_{1}', \ u_{2} = u_{2}', \ u_{3} = u_{3}', \ \phi = \phi', \ \nu_{1,2} = u_{1,2}', \ u_{2,2} = u_{2,2}', \ u_{3,2} = u_{3,2}' \ \text{and} \ \phi_{2} = \phi_{2}' \ \text{on} \ x_{2} = 0, \ \text{for all} \ x_{1} \ \text{and} t.
\]

II. Stress and their derivative w.r.t. \( x_{2} \) are continuous, i.e.

\[
\tau_{12} = \tau_{12}', \ \tau_{22} = \tau_{22}', \ \tau_{23} = \tau_{23}', \ \tau_{12,2} = \tau_{12,2}', \ \tau_{22,2} = \tau_{22,2}', \ \tau_{23,2} = \tau_{23,2}' \ \text{also} \ \tau_{11,2} = \tau_{11,2}', \ \tau_{13,2} = \tau_{13,2}' \ \text{and} \ \tau_{33,2} = \tau_{33,2}' \ \text{on} \ x_{2} = 0, \ \text{for all} \ x_{1} \ \text{and} t
\]

Boundary conditions implies the following equations

\[
\sum_{n=1}^{6} M_{n} = \sum_{n=1}^{6} M_{n}'
\]

\[
\sum_{n=1}^{6} H_{1n} M_{n} = \sum_{n=1}^{6} H_{1n}' M_{n}'
\]

\[
E = F
\]

\[
\sum_{n=1}^{6} H_{2n} M_{n} = \sum_{n=1}^{6} H_{2n}' M_{n}'
\]

\[
\sum_{n=1}^{6} \alpha_{n} M_{n} = \sum_{n=1}^{6} \alpha_{n}' M_{n}'
\]

\[
\sum_{n=1}^{6} \alpha_{n} H_{1n} M_{n} = \sum_{n=1}^{6} \alpha_{n}' H_{1n}' M_{n}'
\]

\[
\eta E = \eta' F
\]

\[
\sum_{n=1}^{6} \alpha_{n} H_{2n} M_{n} = \sum_{n=1}^{6} \alpha_{n}' H_{2n}' M_{n}'
\]
\[
\sum_{n=1}^{6} h_n (-\alpha_n + i \omega H_{1n}) M_n = \sum_{n=1}^{6} h_n' (-\alpha_n' + i \omega H'_{1n}) M_n',
\]
\[
\sum_{n=1}^{6} \{ i \omega (h_{2n} - h_1) - h_4 \alpha_n H_{1n} + \xi H_{2n} \} M_n = \sum_{n=1}^{6} \{ i \omega (h'_{2n} - h'_1) - h'_4 \alpha'_n H'_{1n} + \xi' H'_{2n} \} M_n',
\]
\[
h_2 \eta _1 \eta _2 \eta _3 E = h_2' \eta _1' \eta _2' \eta _3' F,
\]
\[
\sum_{n=1}^{6} h_n \alpha_n (-\alpha_n + i \omega H_{1n}) M_n = \sum_{n=1}^{6} h'_n \alpha'_n (-\alpha'_n + i \omega H'_{1n}) M_n',
\]
\[
\sum_{n=1}^{6} \alpha_n \{ i \omega (h_{2n} - h_1) - h_4 \alpha_n H_{1n} + \xi H_{2n} \} M_n = \sum_{n=1}^{6} \alpha'_n \{ i \omega (h'_{2n} - h'_1) - h'_4 \alpha'_n H'_{1n} + \xi' H'_{2n} \} M_n',
\]
\[
h_3 \eta _1^2 E = h_3' \eta _1'^2 F,
\]
\[
\sum_{n=1}^{6} \alpha_n \{ (i \omega h_3 - \alpha_n (h_2 - h_1) H_{1n}) \} M_n = \sum_{n=1}^{6} \alpha'_n \{ (i \omega h'_3 - \alpha'_n (h'_2 - h'_1) H'_{1n}) \} M_n',
\]
\[
\sum_{n=1}^{6} \alpha_n \{ i \omega (h_{2n} - h_1) - (h_2 - 2h_3) \alpha_n H_{1n} \} M_n = \sum_{n=1}^{6} \alpha'_n \{ i \omega (h'_{2n} - h'_1) - (h'_2 - 2h'_3) \alpha'_n H'_{1n} \} M_n',
\]
\[
\xi \alpha_n H_{2n} M_n = \xi' \alpha'_n H'_{2n} M_n'.
\]

From above set of equations, the four equations containing \(E\) and \(F\) implies that \(E = F = 0\). From remaining twelve equations, for non-trivial solution we have
\[
\det \begin{pmatrix} a_{ij} \end{pmatrix} = 0, \quad i = j = 1, 2, ..., 12. \tag{4.1}
\]

Where
\[
a_{1p} = 1, \quad a_{p+6} = -1; \quad a_{2p} = H_{1p}; \quad a_{2p+6} = -H'_{1p};
\]
\[
a_{3p} = H_{2p}; \quad a_{3p+6} = -H'_{2p};
\]
\[
a_{4p} = \alpha_p; \quad a_{4p+6} = -\alpha'_p;
\]
\[
a_{5p} = \alpha_p H_{1p}; \quad a_{5p+6} = -\alpha'_p H'_{1p};
\]
\[
a_{6p} = \alpha_p H_{2p}; \quad a_{6p+6} = -\alpha'_p H'_{2p};
\]
\[
p = 1, 2, ..., 6;
\]
\[
a_{7p} = h_1 (-\alpha_p + i \omega H_{1p}), \quad p = 1, 2, ..., 6;
\]
\[
a_{8q} = -h'_1 (-\alpha'_q + i \omega H'_{1q}), \quad q = 7, 6, ..., 12.
\]
\[
a_{9p} = \{ i \omega (h_2 - h_1) - h_4 \alpha_p H_{1p} + \xi \alpha_n H_{2n} \}, \quad p = 1, 2, ..., 6;
\]
\[
a_{10q} = -\{ i \omega (h'_2 - h'_1) - h'_4 \alpha'_q H'_{1q} + \xi' \alpha'_n H'_{2q} \}, \quad q = 7, 6, ..., 12.
\]
\[
a_{11p} = \alpha_p \{ (i \omega h_3 - \alpha_n (h_2 - h_1) H_{1p}) \}, \quad p = 1, 2, ..., 6;
\]
\[
a_{12q} = \alpha'_q \{ (i \omega h'_3 - \alpha'_n (h'_2 - h'_1) H'_{1q}) \}, \quad q = 7, 8, ..., 12.
\]
\[
a_{13p} = \alpha_p \{ (i \omega h_3 - \alpha_n (h_2 - h_1) H_{1p}) \}, \quad p = 1, 2, ..., 6;
\]
\[
a_{14q} = \alpha'_q \{ (i \omega h'_3 - \alpha'_n (h'_2 - h'_1) H'_{1q}) \}, \quad q = 7, 8, ..., 12.
\]
\[
a_{15p} = \alpha_p \{ i \omega (h_2 - h_1) - (h_2 - 2h_3) \alpha_p H_{1p} \}, \quad p = 1, 2, ..., 6.
\]
\[
a_{16q} = \alpha'_q \{ i \omega (h'_2 - h'_1) - (h'_2 - 2h'_3) \alpha'_p H'_{1p} \}, \quad q = 7, 8, ..., 12.
\]
\[
a_{17p} = \xi \alpha_p H_{2p}, \quad p = 1, 2, ..., 6.
\]
\[
a_{18q} = -\xi' \alpha'_p H'_{2q}, \quad q = 1, 2, ..., 6.
\]

5. Particular cases
5.1 Stoneley waves
Equation (4.1) is the secular equation for Stoneley waves [4].

Pelagia Research Library
5.2:- Love waves
To investigate the rotational effects on Love waves in a fibre reinforced viscoelastic media of higher order, we replace medium $M_1$ by an infinitely extended horizontal plate of finite thickness $d$ and bounded by two horizontal plane surfaces $x_2 = 0$ and $x_2 = d$. Medium $M$ is semi infinite as in the general case.

The boundary conditions of Love wave are as follows

The displacement component $u_3$ and $\tau_{12}$ between the mediums are continuous, i.e.

$$\begin{align*}
u_3 &= u_3' \quad \text{and} \quad \tau_{23} = \tau_{23}' \quad \text{on} \quad x_2 = 0 \\
\tau_{23}' &= 0 \quad \text{on} \quad x_2 = d, \quad \text{for all} \quad x_1 \text{and} \quad t,
\end{align*}$$

where

$$\begin{align*}
u_3 &= E e^{-\eta_1 x_2} e^{i\omega_1 t}, \\
u_3' &= E' e^{\eta_1 x_2} e^{i\omega_1 t}, \\
\tau_{23} &= \eta e^{\eta_1 x_2} e^{i\omega_1 t}, \\
\tau_{23}' &= 0.
\end{align*}$$

This implies

$$\begin{align*}
E - E' - F' &= 0, \\
\mu_1 \eta_1 E + \mu_2 \eta'_1 E' - \mu_1 \eta'_1 F' &= 0, \\
e^\omega_1 d E' - e^{-\omega_1 d} F' &= 0.
\end{align*}$$

For non trivial solution implies

$$\begin{vmatrix}
1 & -1 & -1 \\
0 & e^{\omega_1 d} & -e^{-\omega_1 d}
\end{vmatrix} = 0,$$

This gives the wave velocity of Love waves propagating in a fiber-reinforced medium. It is interesting to note that voids and non-homogeneity does not affect the velocity of Love waves.

5.3 Rayleigh waves
Rayleigh wave is a special case of the above general surface wave. In this case we consider a model where the medium $M_2$ is replaced by vacuum. Since the boundary $x_2 = 0$ is adjacent to vacuum. It is free from surface traction. So the stress boundary condition in this case may be expressed as

$$\begin{align*}
\tau_{12} &= 0, \quad \tau_{22} = 0, \quad \tau_{12,2} = 0, \quad \tau_{22,2} = 0, \quad \text{also} \quad \tau_{31,2} = 0 \quad \text{and} \quad \tau_{33,2} = 0 \quad \text{on} \quad x_2 = 0, \quad \text{for all} \quad x_1 \text{and} \quad t.
\end{align*}$$

Thus above set of equations reduces to

$$\begin{align*}
\sum_{n=1}^{6} h_1 ( - \alpha_n + i \omega H_{1n} ) M_n &= 0, \\
\sum_{n=1}^{6} \{ i \omega ( h_2 - h_1 ) - h_4 \alpha_n H_{1n} \} M_n &= 0, \\
\sum_{n=1}^{6} h_1 \alpha_n ( - \alpha_n + i \omega H_{1n} ) M_n &= 0, \\
\sum_{n=1}^{6} \alpha_n \{ i \omega ( h_2 - h_1 ) - h_4 \alpha_n H_{1n} - \xi \alpha_n H_{2n} \} M_n &= 0, \\
\sum_{n=1}^{6} \alpha_n \{ i \omega ( h_1 - h_2 ) - ( h_4 - 2 h_5 ) \alpha_n H_{1n} - \xi \alpha_n H_{2n} \} M_n &= 0, \\
\sum_{n=1}^{6} \alpha_n \{ i \omega ( h_2 - h_1 ) - ( h_4 - 2 h_5 ) \alpha_n H_{1n} - \xi \alpha_n H_{2n} \} M_n &= 0.
\end{align*}$$
For non trivial solution
\[ \det(E_{nn}) = 0, \quad n=1,2,\ldots,6 \]  
(5.1)
where
\[ E_{1n} = h_1 \left( -\alpha_n + i\omega H_{1n} \right), \]
\[ E_{2n} = \left\{ i\omega(h_2 - h_1) - h_4 \alpha_n H_{1n} \right\} \]
\[ E_{3n} = h_1 \alpha_n \left( -\alpha_n + i\omega H_{1n} \right), \]
\[ E_{4n} = \alpha_n \left\{ i\omega(h_2 - h_1) - h_4 \alpha_n H_{1n} - \xi \alpha_n H_{2n} \right\} \]
\[ E_{5n} = \alpha_n \left( i\omega(h_3 - h_1)(h_2 - h_1)H_{1n} - \xi \alpha_n H_{2n} \right) \]
\[ E_{6n} = \alpha_n \left\{ i\omega(h_2 - h_1) - (h_4 - 2h_5) \alpha_n H_{1n} - \xi \alpha_n H_{2n} \right\} \]
Equation (5.1) is the secular equation for Rayleigh wave.

**DISCUSSION AND CONCLUSION**

Very few researchers did work in that field because of complicated nature of the governing equations of the fibre-reinforced anisotropic with voids. The method used in this study provides a quite successful in dealing with such problems. This method gives exact solutions in the fibre-reinforced anisotropic elastic media without any assumption. Special cases only for Stonely, Love and Rayleigh waves were considered.

It is observed that in the case of homogeneous and without voids only one mode propagate in the medium but in the case of non-homogeneous without voids two mode exist. In the case of homogeneous with voids, three modes exist. In non-homogeneous with voids maximum six modes can propagate and it is depend upon the nature of material.

**REFERENCES**