ABSTRACT

The pulsatile flow of Herschel – Bulkley fluid through an inclined multiple stenoses artery with periodic body acceleration has been investigated in this paper. Assuming the stenoses to be mild, the nonlinear equations governing the flow are solved using perturbation technique. Analytical expressions are obtained for axial velocity, plug velocity, wall shear stress and flow rate. Their variations with different flow parameters are plotted in figures. It is noticed that the velocity increases as body acceleration increases but it decreases as yield stress increases and wall shear stress increases as body acceleration increases.

Keywords: Pulsatile flow, Body acceleration, Herschel – Bulkley fluid, stenosed artery.

INTRODUCTION


In the present investigation an effort has been made to study the pulsatile flow of Herschel – Bulkley fluid through an inclined multiple stenoses artery with non-uniform cross-section subject to periodic body acceleration assuming that the stenoses are mild. Analytical expressions for axial velocity and flow rate have been derived and the effects of various parameters on these flow variables have been studied.

**MATHEMATICAL FORMULATION**

\[
\overline{R}(\zeta) = \begin{cases} 
R_0 & : 0 \leq \zeta \leq d_1, \\
R_0 - \frac{\delta_1}{2} \left( 1 + \cos \frac{2\pi}{L_1} \left( \zeta - d_1 - \frac{L_1}{2} \right) \right) & : d_1 \leq \zeta \leq d_1 + L_1, \\
R_0 - \frac{\delta_1}{L_1} \left( 1 + \cos \frac{2\pi}{L_1} \left( \zeta - B_1 \right) \right) & : d_1 + L_1 \leq \zeta \leq B_1 - \frac{L_2}{2}, \\
R^* \left( \zeta \right) - \frac{\delta_2}{2} \left( 1 + \cos \frac{2\pi}{L_2} \left( \zeta - B_1 \right) \right) & : B_1 - \frac{L_2}{2} \leq \zeta \leq B_1, \\
R^* \left( \zeta \right) & : B_1 \leq \zeta \leq B_1 + \frac{L_2}{2}, \\
& : B_1 + \frac{L_2}{2} \leq \zeta \leq B.
\end{cases}
\]

(1)

The following restrictions for mild stenoses [10] are supposed to be satisfied:
\[\delta_i \ll \min \left( R_0, R_{\text{out}} \right),\]
\[\delta_i \ll L_i, \quad \text{where } R_{\text{out}} = R(\zeta) \text{ at } \zeta = B.\]

Here \( L_i \) and \( \delta_i (i = 1, 2) \) are the lengths and maximum heights of two stenoses (the suffixes 1 and 2 refer to the first and second stenosis respectively).

The pressure gradient and body acceleration are given by:
\[\frac{\partial \overline{p}}{\partial \zeta} = A_0 + A_1 \cos(\omega_p \tau),\]  
(2)
\[G(\tau) = a_0 \cos(\omega_h \tau + \phi)\]  
(3)
Where $A_0$ and $A_1$ are pressure gradient of steady flow and amplitude of oscillatory part respectively, $a_0$ is the amplitude of body acceleration, $\omega_p = 2\pi f_p$, $\omega_b = 2\pi f_b$ with $f_p$ is the pulse frequency and $f_b$ is body acceleration frequency, $\phi$ is the phase angle of body acceleration with respect to the pressure gradient and $t$ is time.

Figure: 1 Geometry of an inclined tube with multiple stenoses

The governing equation of motion for flow in cylindrical polar coordinates can be written in the form:

$$\frac{\partial \bar{u}}{\partial t} = -\frac{\partial \bar{p}}{\partial \bar{r}} - \frac{1}{\bar{r}} \frac{\partial}{\partial \bar{r}} (\bar{r} \bar{\tau}) + G(\bar{T}) + \bar{p} \bar{g} \sin \beta$$

$$\frac{\partial \bar{u}}{\partial t} = A_0 + A_1 \cos(\omega_p \bar{T}) - \frac{1}{\bar{r}} \frac{\partial}{\partial \bar{r}} (\bar{r} \bar{\tau}) + a_0 \cos(\omega_b \bar{T} + \phi) + \bar{p} \bar{g} \sin \beta$$

(4)

$$\frac{\partial \bar{p}}{\partial \bar{r}} = 0$$

(5)

Where $\bar{r}, \bar{z}$ denote the radial and axial coordinates respectively and $\bar{\rho}$ denote density, $\bar{u}$ axial velocity of blood, $\bar{T}$ time, $\bar{p}$ pressure and $\bar{\tau}$ the shear stress and $\beta$ be the small angle of inclination, $g$ is acceleration due to gravity. For Herschell-Bulkley fluid the relation between shear stress and shear rate is given by

$$\bar{\tau} = \bar{\mu}_H \left( \frac{-\partial \bar{u}}{\partial \bar{r}} \right)^n + \bar{\tau}_H \text{ if } \bar{\tau} \geq \bar{\tau}_H$$

(6)

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\[ \frac{\partial \bar{u}}{\partial \bar{r}} = 0 \quad \text{if} \quad \bar{\tau} < \bar{\tau}_H \quad (7) \]

Where \( \bar{u} \) is the total velocity, \( \bar{\tau}_H \) is the yield stress, \( n \) is the power law index, and \( \bar{\mu}_H \) is the coefficient of viscosity for Herschell-Bulkley fluid.

When \( \bar{\tau} < \bar{\tau}_H \), i.e., the shear stress is less than the yield stress, there is a core region which flows as a plug and Eq. (7) corresponds to vanishing velocity gradient in that region. However, the fluid behavior is indicated whenever \( \bar{\tau} > \bar{\tau}_H \).

The boundary conditions are:

\[ \bar{\tau} \text{ is finite at } \bar{r} = 0 \quad (8) \]
\[ \bar{u} = 0 \text{ at } \bar{r} = \bar{R}(\bar{\tau}) \quad (9) \]

Introducing the non-dimensional variables:

\[ u = \frac{\bar{u}}{A_0R_0^2/4\mu_0}, \quad z = \frac{\bar{z}}{R_0}, \quad t = \omega_p\bar{r}, \quad \delta = \frac{\bar{\delta}}{R_0}, \quad \tau = \frac{\bar{\tau}}{A_0R_0/2} \]

\[ \tau_H = \frac{\bar{\tau}_H}{A_0R_0/2}, \quad R(z) = \frac{\bar{R}(\bar{\tau})}{R_0}, \quad r = \frac{\bar{r}}{R_0}, \quad d_l = \frac{\bar{d}_l}{B}, \quad L_l = \frac{\bar{L}_l}{B} \]

\[ L_2 = \frac{\bar{L}_2}{B}, \quad B_1 = \frac{\bar{B}_1}{B}, \quad a = \frac{a_0}{A_0}, \quad e = \frac{A_1}{A_0}, \quad \omega = \frac{\omega_b}{\omega_p} \]

\[ \mu_0 = \bar{\mu}_H \left( \frac{2}{R_0A_0} \right)^{n-1}, \quad F = \frac{A_0}{4\rho g} \quad (10) \]

The non-dimensional momentum equation (4) becomes

\[ \alpha^2 \frac{\partial u}{\partial t} = 4(1 + e\cos t) - \frac{2}{r} \frac{\partial}{\partial r} \left( r\tau \right) + 4a\cos(\omega t + \phi) + \frac{\sin \beta}{F} \quad (11) \]

Where \( \alpha^2 = \frac{\omega R^2}{\mu / \rho} \), \( \alpha \) is Womersley frequency parameter.

Equations (6) and (7) can be written as

\[ (\tau - \tau_H)^{1/n} = \left( -\frac{1}{2} \frac{\partial u}{\partial r} \right) \text{if} \ \tau > \tau_H \quad (12) \]

\[ \frac{\partial u}{\partial r} = 0 \quad \text{if} \ \tau < \tau_H \quad (13) \]

The boundary conditions (equations 8 and 9) reduce to
\[ \tau \text{ is finite at } r = 0 \]  
\[ u = 0 \text{ at } r = R(z) \]  
\[ \frac{1}{2} \delta \left( 1 + \cos \frac{2\pi}{L_1} \left( z - d_1 - \frac{L_1}{2} \right) \right) \]  
\[ : 0 \leq z \leq d_1, \]  
\[ 1 \]  
\[ : d_1 \leq z \leq d_1 + L_1, \]  
\[ \frac{1}{2} \delta \left( 1 + \cos \frac{2\pi}{L_2} \left( z - B_1 \right) \right) \]  
\[ : B_1 - \frac{L_2}{2} \leq z \leq B_1, \]  
\[ R(z) = \]  
\[ \frac{1}{2} \delta \left( 1 + \cos \frac{2\pi}{L_2} \left( z - B_1 \right) \right) \]  
\[ : B_1 \leq z \leq B_1 + \frac{L_2}{2}, \]  
\[ R^* (z) = \]  
\[ : B_1 + \frac{L_2}{2} \leq z \leq B. \]  

**METHOD OF SOLUTION**

On using perturbation method, the velocity \( u \) and shear stress \( \tau \) are expanded as follows in terms of \( \alpha^2 \) (where \( \alpha^2 \ll 1 \))

\[ u(z, r, t) = u_0(z, r, t) + \alpha^2 u_1(z, r, t) + \ldots \ldots \]  
\[ (17) \]  
\[ \tau(z, r, t) = \tau_0(z, r, t) + \alpha^2 \tau_1(z, r, t) + \ldots \ldots \]  
\[ (18) \]  
\[ u_p(z, r, t) = u_{0p}(z, r, t) + \alpha^2 u_{1p}(z, r, t) + \ldots \ldots \]  
\[ (19) \]  
\[ R_p(z, r, t) = R_{0p}(z, r, t) + \alpha^2 R_{1p}(z, r, t) + \ldots \ldots \]  
\[ (20) \]  

Substituting (17) and (18) in equation (11) and equating the constant term and \( \alpha^2 \) term we get

\[ \frac{\partial}{\partial r}(r \tau_0) = 2r \left\{ (1 + e \cos t) + a \cos(\alpha \theta + \phi) + \frac{\sin \beta}{4F} \right\} \]  
\[ (21) \]  
\[ \frac{\partial u_0}{\partial t} = -2 \frac{\partial}{r \partial r}(r \tau_1) \]  
\[ (22) \]  

Integrate equation (21) and using boundary condition (14)

\[ \tau_0 = f(t) \, r \]  
\[ (23) \]  

Where

\[ f(t) = 1 + e \cos t + a \cos(\alpha \theta + \phi) + \frac{\sin \beta}{4F} \]

Substituting (17) and (18) in (12)
\[- \frac{\partial u_0}{\partial r} = 2\tau_0^{k-1}(\tau_0 - k\tau_H) \]  
(24)

\[- \frac{\partial u_1}{\partial r} = 2k\tau_0^{k-2}\tau_1(\tau_0 - (k-1)\tau_H) \]  
(25)

Where \( k = \frac{1}{n} \)

Integrating equation (24) using the relation (23) and the boundary condition (15) we obtain

\[ u_0 = A_1\left(R^{k+1} - r^{k+1}\right) + A_2\left(R^k - r^k\right) \]  
(26)

Where \( A_1 = \frac{2f^k(t)}{n+1}, \quad A_2 = -2f^{k-1}(t)\tau_H \)

The plug core velocity \( u_{0p} \) can be obtained from equation (26) as

\[ u_{0p} = A_1\left(R^{k+1} - R_{0p}^{k+1}\right) + A_2\left(R^k - R_{0p}^k\right) \]  
(27)

Neglecting the terms of \( O(\alpha^2) \) and higher powers of \( \alpha \) in equation (20) \( R_{0p} \) can be obtained from (23) as

\[ R_{0p} = \frac{\tau_H}{f(t)} \]  
(28)

Using equation (22) we get the solution for \( \tau_1 \) as,

\[ \tau_1 = a_4 r + a_6 r^{k+1} + a_7 r^{k+2} \]  
(29)

Where \( a_1 = -\frac{nR^{k+1}}{2(k+1)}, \quad a_2 = \frac{(k-1)\tau_H R^k}{2}, \quad a_3 = -\frac{(k-1)\tau_H}{k+2}, \quad a_4 = \frac{k}{(k+1)(k+3)}, \quad a_5 = (a_1 + a_2)f^{k-1}(t)f'(t), \quad a_6 = a_3f^{k-2}(t)f'(t), \quad a_7 = a_4f^{k-1}(t)f'(t) \)

Similarly using equations (25) and (29) we can obtain the solution for \( u_1 \) as

\[ u_1 = \{ (b_1 + b_2)R^{k+2} - b_2r^{2k+2} \} + \{ (b_3 + b_4) - (b_5 + b_6) \} R^{k+1} - (b_7 - b_8)r^{2k+1} \]  
(30)

\[ -b_3 R^{2k} + b_{10}r^{2k} - b_1R^{k+1}r^{k+1} - b_2R^kr^{k+1} + b_3R^{k+1}r^k + b_4R^k r^k \]

\[ u_{1p} = \{ (b_1 + b_2)R^{k+2} - b_3R_{0p}^{2k+2} \} + \{ (b_3 + b_4) - (b_5 + b_6) \} R^{k+1} - (b_7 - b_8)R_{0p}^{2k+1} \]  
(31)

\[ -b_3 R^{2k} + b_{10}R_{0p}^{2k} - b_1R^{k+1}R_{0p}^{k+1} - b_2R^{k}R_{0p}^{k+1} + b_3R^{k+1}R_{0p}^k + b_4R^k R_{0p}^k \]
Using equations (17) and (18) the total velocity distribution and shear stress can be written as

\[ u = A_1 \left( R^{k+1} - R^k \right) + A_2 \left( R^k - R^k \right) + \alpha^2 \left[ \left\{ (b_1 + b_2) R^{k+2} - b_2 r^{2k+2} \right\} + \left\{ (b_3 + b_4) - (b_5 + b_6) \right\} R^{2k+4} - (b_7 - b_8) r^{2k+1} \right] \]

\[ = f(t) R + \alpha^2 \left\{ a_5 R + a_6 R^{k+1} + a_7 R^{k+2} \right\} \]

The second approximation plug core radius \( R_{1p} \) can be obtained by neglecting terms of \( O(\alpha^4) \) and higher powers of \( \alpha \) in equation (20) as

\[ R_{1p} = \frac{\tau_1(R_{0p})}{f(t)} \]

With the help of equations (20), (28) and (35), \( R_p \) can be given by

\[ R_p = \frac{\tau_H}{f(t)} + \frac{\alpha^2}{f(t)} \left( a_5 R_{0p} + a_6 R_{0p}^{k+1} + a_7 R_{0p}^{k+2} \right) \]

The volumetric flow rate \( Q \) is given by

\[ Q = 4 f(z) \int_0^R u(z, r, t) dr \]
\[ Q = 4 \left[ A_1 R^{k+3} \frac{(k+1)}{2(k+3)} + A_2 R^{k+2} \frac{k}{2(k+2)} + \alpha^2 \left\{ R^{k+4} \frac{(b_1 + b_2)}{2} \right. \right. \]

\[ \left. - R^{2k+4} \left( \frac{b_2}{2k+4} + \frac{b_{12}}{k+3} \right) + R^{2k+3} \left( \frac{b_3 + b_4 - b_5 - b_6}{2} \right) \right. \]

\[ \left. - \frac{b_3 - b_5 - b_{12} + b_{13}}{2k+3} \right\} + R^{2k+2} \left( \frac{-b_9}{2} + \frac{b_{10}}{2k+2} + \frac{b_{14}}{k+2} \right) \right] \]

(37)

Figure 2: Variation of axial velocity with radial distance \( r \) for \( \tau_{H} = 0.1, a = 2, \omega = 1, \phi = 0.2, \beta = 0.1 \)

Figure 3: Variation of axial velocity with radial distance \( r \) for \( \tau_{H} = 0.1, e = 2, \omega = 1, \phi = 0.2, \beta = 0.1 \)
Figure 4: Variation of axial velocity with radial distance $r$ for $\tau_H = 0.1$, $a = 2$, $\omega = 1$, $\phi = 0.2$, $\beta = 0.1$

Figure 5: Variation of axial velocity with radial distance $r$ for $\tau_H = 0.1$, $a = 2$, $\omega = 1$, $e = 2$, $\beta = 0.1$
Figure 6: Variation of axial velocity with radial distance $r$ for $\tau_H = 0.1$, $a = 2$, $\omega = 1$, $\phi = 0.2$, $e = 2$

Figure 7: Variation of axial velocity with radial distance $r$ for $e = 2$, $a = 2$, $\omega = 1$, $\phi = 0.2$, $\beta = 0.1$
RESULTS AND DISCUSSION

The velocity profile for the pulsatile flow of Herschel–Bulkley fluid through an inclined multiple stenoses artery with periodic body acceleration is computed by using (32) for different values of parameter $e$, body acceleration parameter $a$, time $t$, phase angle $\phi$, inclination angle $\beta$, yield stress $\tau_H$ have been shown through figures 2-7. Figure 2 shows that the variation of velocity profile for different values of parameter $e$. It can be noted here that as the parameter $e$ increases the velocity profile increases. In the presence of body acceleration, velocity increases rapidly. As the body acceleration increases, the plug region shrinks and hence more flow takes place (figure 3). It can easily be seen from figures 4 & 5 that an increase in the time $t$ and phase angle $\phi$ leads to decrease in the velocity profile. From figures 6 and 7 it can be observed that an increase in the inclination angle $\beta$ and yield stress $\tau_H$ lead to an increase in velocity profile. Variation of wall shear stress with time $t$ is presented in figure 8. From this figure, it can be clearly observed that for any value of body acceleration parameter $a$, wall shear stress gradually decreases as time $t$ increases until it attains its minimum at $t = 180^\circ$, wherefrom it gradually increases with time and reaches its approached magnitude at $t = 360^\circ$.

CONCLUSION

The present study deals with a theoretical investigation of the characteristics of the pulsatile flow of blood through an inclined multiple stenoses artery with periodic body acceleration. Blood is represented by Herschel–Bulkley fluid model. Using appropriate boundary conditions, analytical expressions for the velocity and flow rate have been obtained. It is clear from the above result and discussions that the body acceleration effects largely on the axial velocity of blood flow.

A proper understanding of interactions of body acceleration with blood flow in presence of inclination could be useful in the diagnosis and therapeutic treatment of some health problems (joint pain, vision loss and vascular disorder) to better design of protective pads and machines.
Hence from all the above discussions we can conclude that a careful choice of the values of the parameters of body acceleration, yield stress and inclination angle will affect the flow characteristics and hence can be utilised for medical and engineering applications.

REFERENCES