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On Some Length-Biased Weighted Weibull Distribution

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ABSTRACT

In this paper, we develop the length-biased form of the weighted Weibull distribution (WWD) named as length-biased weighted Weibull distribution (LBWWD). Well known distributions are generated by expanding suitable function of the parameters. Shape of the distribution is studied in detail. Various properties of length-biased WWD are discussed. Newby's method along with method of moment has been used to estimate the parameters of the length-biased WWD. To justify the use of LBWWD; LBWWD is fitted to 30 consecutive year's data from 1981 - 2010 of June rainfall (in mm) of Tezpur, Assam, India.

Keywords: Length-biased, Weibull distribution, weighted Weibull distribution, moments.

INTRODUCTION

The Weibull distribution is a member of the family of extreme value distributions. These distributions are the limit distributions of the smallest or the greatest value, respectively, in a sample with sample size $n \to \infty$. The Weibull distribution includes the exponential and the Rayleigh distributions as special cases. The use of the distribution in reliability and quality control work was advocated by many authors following Kao [12], [13] and Berrettoni [1]. Malik [18] and Franck [6] have assigned some simple physical meanings and interpretations for the Weibull distribution, thus providing natural applications of this distribution in reliability problems particularly dealing with wearing styles. Johnson *et al*., [11] has provided an excellent review of applications on Weibull distribution. The usefulness and applications of parametric distributions including Weibull, Rayleigh are seen in various areas including reliability, renewal theory, and branching processes which can be seen in papers by several authors including Patil and Rao [21], Gupta and Kirmani [10], Gupta and Keating [9], Oluyede [20] and in references therein.

Distribution of the type $f^w(x) = \frac{w(x)f(x)}{W}$ where $W = \int w(x)f(x)$, with an arbitrary non-negative function $w(x)$ which may exceed unity was introduced by Rao [24]. He have given practical examples where $w(x) = x$ or x^α were appropriate. He called distributions with arbitrary weight w(x) weighted distributions. First introduced by Fisher [5] to model ascertainment bias, weighted distributions were later formalized in a unifying theory by Rao [24]. Weighted distributions have numerous applications. Weighted distribution concept of Patil, Rao and Zelen [22] can be traced to the study of the effect of methods of ascertainment estimation of frequencies by Fisher [5]. Van Deusen [27] arrived at size-biased distribution theory independently and applied it to fitting distributions of diameter at breast height (DBH) data arising from horizontal point sampling (HPS) (Grosenbaugh [8]) inventories. Subsequently, Lappi and Bailey [16] used weighted distributions to analyse HPS diameter increment data. Weighted distributions were used by Magnussen *et al*., [17] to recover the distribution of canopy heights from airborne laser scanner measurements. In ecology, Dennis and Patil [4] used stochastic differential equations to arrive at a weighted gamma distribution as the stationary probability density function for a stochastic population model with predation effects. In fisheries, Taillie *et al*., [26] modeled populations of fish stocks using weighted distributions.

When the probability of observing a positive-valued random variable is proportional to the value of the variable the resultant is length-biased distribution. The length-biased form of the weighted Weibull distribution is considered because there may be situations when the lifetimes of a given sample of objects is weighted Weibull but the objects may not have the same chance of being selected but each one is selected according to its length or life length then the resulting distribution is not weighted Weibull but length-biased weighted Weibull. A table for some basic distributions and their length-biased forms is given by Patil and Rao [21] such as Lognormal, Gamma, Pareto, Beta distribution. Khatree [15] presented a useful result by giving a relationship between the original random variable *X* and its length-biased version *Y* when *X* is either Inverse Gaussian or Gamma distribution. Das *et al*., [2] developed the length biased distribution (LBD) of weighted Inverse Gaussian distribution (WIGD). Das and Roy [3] developed the length-biased form of the Weighted Generalized Rayleigh distribution (WGRD) known as length-biased Weighted Generalized Rayleigh distribution.

MATERIALS AND METHODS

30 consecutive years data from 1981 - 2010 of June rainfall (in mm) of Tezpur, Assam, India is considered for the study. The data has been collected from Regional Meteoroligical Centre, LGBI Airport, Guwahati [7].

Rao [24] introduced the concept of a weighted distribution, let x be a nonnegative random variable (rv) with probability density function (pdf) f(x). Let the weight function be w(x) which is a non-negative function. Then the weighted density function f(x) is obtained as

$$f(x) = \frac{w(x)f(x)}{\int_{-\infty}^{\infty} w(x)f(x)} \qquad x > 0 \tag{1}$$

assuming that $E(X) = \int_{-\infty}^{\infty} w(x)f(x) < \infty$ i.e the first moment of w(x) exists.

By taking weight as w(x) = x we obtain length-biased distribution.

Derivation of length-Biased weighted Weibull distribution
The probability density function of the Weibull random variable X is then

$$f(x) = \frac{c}{\alpha}\left(\frac{x-\xi_0}{\alpha}\right)^{c-1} \exp\left[-\left(\frac{x-\xi_0}{\alpha}\right)^{c}\right], x > \xi_0, c > 0, \alpha > 0$$

The two-parameter Weibull pdf is obtained by setting $\xi_0 = 0$ and is given by

$$f(x) = \frac{c}{\alpha}\left(\frac{x}{\alpha}\right)^{c-1} exp\left[-\left(\frac{x}{\alpha}\right)^{c}\right], x > 0, c > 0, \alpha > 0 \quad (2)$$

Let

$$w(x) = x^{c\beta} \quad (3)$$

Substituting equation (2) and (3) in (1) we get

$$h(x) = \frac{c}{\alpha^{c+c\beta}\Gamma(\beta+1)} x^{c+c\beta-1} exp\left[-\left(\frac{x}{\alpha}\right)^{c}\right], x > 0, c > 0, \alpha > 0, \beta > 0 \quad (4)$$

The density function in (4) is known as weighted Weibull distribution (WWD)

Again substituting w(x) = x in (1) and using the density function of (4) we obtain the density function of the length-biased weighted Weibull distribution (LBWWD) of the form

$$k(x) = \frac{c}{\alpha^{c+c\beta+1}\Gamma(\beta+\frac{1}{c}+1)} x^{c+c\beta} exp\left[-\left(\frac{x}{\alpha}\right)^{c}\right], x > 0, c > 0, \alpha > 0, \beta > 0 \quad (5)$$

The cumulative distribution function (cdf) of (5) is defined as

$$K(x) = \frac{\gamma\left(\beta + \frac{1}{c} + 1, \left(\frac{x}{\alpha}\right)^{c}\right)}{\Gamma(\beta + \frac{1}{c} + 1)}$$

where $\gamma(a, x) = P(a, x)\Gamma(a) = \int_0^{\infty} e^{-t}t^{a-1}dt$, is the incomplete gamma function.

Some particular cases of LBWWD:

1. Putting $\beta = -\frac{1}{c}$, $c = c$ and $\alpha = \alpha$ in (5), we get Weibull distribution
2. $$k(x) = \frac{c}{\alpha}\left(\frac{x}{\alpha}\right)^{c-1} exp\left[-\left(\frac{x}{\alpha}\right)^{c}\right], x > 0, c > 0, \alpha > 0$$
3. By substituting, $\beta = \beta - \frac{1}{c}$, $c = c$ and $\alpha = \alpha$ in (5), we get weighted Weibull distribution

$$k(x) = \frac{c}{\alpha^{c+c\beta}\Gamma(\beta+1)}\left(\frac{x}{\alpha}\right)^{c+c\beta-1} exp\left[-\left(\frac{x}{\alpha}\right)^{c}\right], x > 0, c > 0, \alpha > 0, \beta > 0$$

4. In case, $c = 2$, $\beta = 0$, $\alpha = \sqrt{\alpha}$ and multiplying a constant $k = 2$ in (5) reduces to
5.

$$k(x) = \frac{2^{\frac{3}{2}}}{\sqrt{\pi\alpha^3}} x^2\, exp\left[-\frac{x^2}{2\alpha^2}\right], x > 0, c > 0, \alpha > 0$$

this density function is known in the statistics and physics literatures as "Maxwell- Boltzmann density function."

6. Substituting $c = 1$, $\alpha = 1$ and $\beta = \beta - 2$ in (5), we obtain the density function of the Gamma distribution
7.

$$k(x) = \frac{1}{\Gamma(\beta)} x^{\beta-1} e^{-x}, x > 0, \beta > 0$$

8. Putting $c = 2$, $\beta = \frac{N}{2} - \frac{3}{2}$, (5) yields Generalized Rayleigh distribution
9.

$$k(x) = \frac{2}{(2\alpha^2)^{N/2}\Gamma(N/2)} x^{N-1} exp\left(-\frac{x^2}{2\alpha^2}\right), \quad x > 0, \alpha > 0$$

This generalized form of the Rayleigh distribution is also referred in literature as the chi distribution with N degrees of freedom and scale parameter α.

Shape of LBWWD

The shape of (5) can be sorted by studying the function defined over $[0, \infty]$ and the behavior of its derivative.

Limits of the function

1. $$\boldsymbol{lim_{x\to 0} k(x)} = \frac{c}{\alpha^{c+c\beta+1}\Gamma(\beta+\frac{1}{c}+1)} lim_{x\to 0} x^{c+c\beta} exp\left[-\left(\frac{x}{\alpha}\right)^{c}\right]$$

$$\because lim_{x\to 0} x^{c+c\beta} exp\left[-\left(\frac{x}{\alpha}\right)^{c}\right] = 0$$

$$\therefore \boldsymbol{lim_{x\to 0} k(x) = 0}$$

2. $$\boldsymbol{lim_{x\to \infty} k(x)} = \frac{c}{\alpha^{c+c\beta+1}\Gamma(\beta+\frac{1}{c}+1)} lim_{x\to \infty} x^{c+c\beta} exp\left[-\left(\frac{x}{\alpha}\right)^{c}\right]$$

$$lim_{x\to \infty} exp\left[-\left(\frac{x}{\alpha}\right)^{c}\right] = 0$$

$$\therefore \boldsymbol{lim_{x\to 0} k(x) = 0}$$

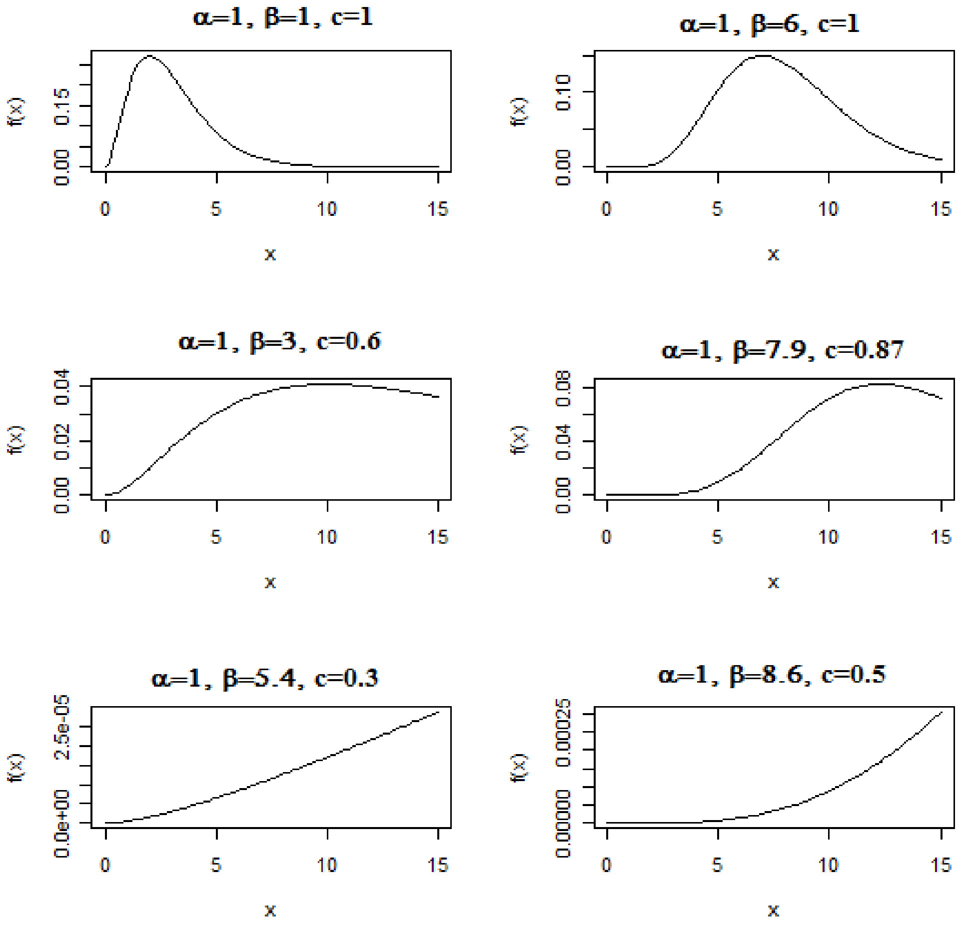

Fig 1: Length-biased weighted Weibull densities

The first and second derivatives of (5) with respect to *x* are obtained as follows:
Taking logarithm on *k*(*x*) in (5) we get

$$ln[k(x)] = ln(c) - ln\left[\Gamma(\beta + \frac{1}{c} + 1)\right] - (c + c\beta + 1)ln(\alpha) + (c + c\beta)ln(x) - \left(\frac{x}{\alpha}\right)^c$$

Differentiating w.r.t *x* gives

$$\frac{d}{dx}ln[k(x)] = \frac{c + c\beta - \frac{c}{\alpha^c}x^c}{x}$$

Equating the above derivative to 0 leads to, $x_0 = \alpha[\beta + 1]^{\frac{1}{c}}$

The derivative equals 0 at *x*0, positive for values of *x* that exceed x_0 and negative otherwise. The second derivative of *k*(*x*) w.r.t *x* yields.

$$\frac{d^2}{dx^2} = -\frac{1}{x^2}\left[c + c\beta + \frac{c(c-1)x^c}{\alpha^c}\right]$$

For all values of x the quantity is negative. Figure (1), suggests for different values of c and β with α = 1 the shape of equation (5). Here α = 1 since it is a scale parameter and hence doesn't influence the shape of the curve.

Properties of LBWWD

Reliability function of LBWWD:

The reliability function of the LBWWD also known as the survivor function is defined as

$$R(x) = 1 - \frac{\gamma\left(\beta + \frac{1}{c} + 1,\ \left(\frac{x}{\alpha}\right)^c\right)}{\Gamma(\beta + \frac{1}{c} + 1)} \tag{6}$$

Table (1) contains the values of survival function (6). Looking at this table we can see that the survival probability of the distribution increases with increase in the value of c for a holding x and α and β at a fixed level. Further, from the table we can see that; for fixed c, α and β; the survival probability decreases with increase in x.

Table 1: Survival function of LBWWD

α=1, β=1							
	c						
x	1.0	1.2	1.4	1.6	1.8	2.0	2.5
0.1	0.9999227	0.9999549	0.9999768	0.9999889	0.9999949	0.9999978	0.9999997
0.2	0.9994258	0.9995518	0.9996914	0.9998030	0.9998803	0.9999296	0.9999829
0.3	0.9982003	0.9983352	0.9986408	0.9989713	0.9992590	0.9994841	0.9998099
0.4	0.9960368	0.9958748	0.9962044	0.9967599	0.9973667	0.9979312	0.9989699
0.5	0.9928062	0.9918189	0.9917607	0.9922917	0.9931279	0.9940747	0.9962728
0.6	0.9884424	0.9859138	0.9847696	0.9846815	0.9853020	0.9863489	0.9896329
0.7	0.9829292	0.9780076	0.9748269	0.9731616	0.9726731	0.9730413	0.9761167
0.8	0.9762887	0.9680439	0.9616906	0.9571781	0.9542442	0.9525901	0.9523640
0.9	0.9685715	0.9560503	0.9452893	0.9364523	0.9293966	0.9238872	0.9153569
1.0	0.9598493	0.9421251	0.9257130	0.9110040	0.8979904	0.8865189	0.8633480

Hazard function of LBWWD:

The hazard function of LBWWD is given by

$$h(x) = \frac{cx^{c+c\beta} exp\left[-\left(\frac{x}{\alpha}\right)^c\right]}{\alpha^{c+c\beta+1}\left[\Gamma\left(\beta+\frac{1}{c}+1\right) - \gamma\left(\beta+\frac{1}{c}+1,\ \left(\frac{x}{\alpha}\right)^c\right)\right]}$$

Moments of LBWWD:
If *X* follows LBWWD with parameters *c*, *β* and *α*, then the rth moment of *X*, say μ_r', is given as

$$\mu_r' = \frac{\alpha^r \Gamma\left(\beta + \frac{r}{c} + \frac{1}{c} + 1\right)}{\Gamma\left(\beta + \frac{1}{c} + 1\right)} = \frac{\alpha^r \Gamma_{(r+1)\beta}}{\Gamma_{1\beta}}$$

where $\Gamma_{(r+1)\beta} = \Gamma\left(\beta + \frac{r}{c} + \frac{1}{c} + 1\right)$ and $\Gamma_{r\beta} = \Gamma\left(\beta + \frac{r}{c} + 1\right)$, r= 1, 2, 3,…

For the case, r = 1, 2, 3, 4 we have
$\mu_1' = \frac{\alpha\Gamma_{2\beta}}{\Gamma_{1\beta}}$, $\mu_2' = \frac{\alpha^2\Gamma_{3\beta}}{\Gamma_{1\beta}}$, $\mu_3' = \frac{\alpha^3\Gamma_{4\beta}}{\Gamma_{1\beta}}$, $\mu_4' = \frac{\alpha^3\Gamma_{5\beta}}{\Gamma_{1\beta}}$

The first four central moments of LBWWD are given by

$$\mu_1 = 0$$

$$\mu_2 = \alpha^2\left[\frac{\Gamma_{3\beta}}{\Gamma_{1\beta}} - \frac{\Gamma^2{}_{2\beta}}{\Gamma^2{}_{1\beta}}\right]$$

$$\mu_3 = \alpha^3\left[\frac{\Gamma_{4\beta}}{\Gamma_{1\beta}} - \frac{3\Gamma_{3\beta}\Gamma_{2\beta}}{\Gamma^2{}_{1\beta}} + \frac{2\Gamma^3{}_{2\beta}}{\Gamma^3{}_{1\beta}}\right]$$

$$\mu_4 = \alpha^4\left[\frac{\Gamma_{5\beta}}{\Gamma_{1\beta}} - \frac{4\Gamma_{4\beta}\Gamma_{2\beta}}{\Gamma^2{}_{1\beta}} + \frac{6\Gamma_{3\beta}\Gamma^2{}_{2\beta}}{\Gamma^3{}_{1\beta}} - \frac{3\Gamma^4{}_{2\beta}}{\Gamma^4{}_{1\beta}}\right]$$

Coefficient of variation of LBWWD is of the form

$$cov^{**} = \Gamma_{1\beta}\Gamma^{-1}{}_{2\beta}\sqrt{\left[\frac{\Gamma_{3\beta}}{\Gamma_{1\beta}} - \frac{\Gamma^2{}_{2\beta}}{\Gamma^2{}_{1\beta}}\right]}$$

cov^{**}, coefficient of variation of LBWWD

Measure of skewness of LBWWD is given by

$$\gamma_1 = \frac{\left[\frac{\Gamma_{4\beta}}{\Gamma_{1\beta}} - \frac{3\Gamma_{3\beta}\Gamma_{2\beta}}{\Gamma^2{}_{1\beta}} + \frac{2\Gamma^3{}_{2\beta}}{\Gamma^3{}_{1\beta}}\right]}{\left[\frac{\Gamma_{3\beta}}{\Gamma_{1\beta}} - \frac{\Gamma^2{}_{2\beta}}{\Gamma^2{}_{1\beta}}\right]^{\frac{3}{2}}}$$

Measure of skewness of LBWWD is given by

$$\gamma_2 = \frac{\left[\frac{\Gamma_{5\beta}}{\Gamma_{1\beta}} - \frac{4\Gamma_{4\beta}\Gamma_{2\beta}}{\Gamma^2{}_{1\beta}} + \frac{6\Gamma_{3\beta}\Gamma^2{}_{2\beta}}{\Gamma^3{}_{1\beta}} - \frac{3\Gamma^4{}_{2\beta}}{\Gamma^4{}_{1\beta}}\right]}{\left[\frac{\Gamma_{3\beta}}{\Gamma_{1\beta}} - \frac{\Gamma^2{}_{2\beta}}{\Gamma^2{}_{1\beta}}\right]^2} - 3$$

The moment generating function of LBWWD is given by

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \frac{\alpha^r \Gamma_{(r+1)\beta}}{\Gamma_{1\beta}}$$

The cumulant generating function of LBWWD is given by

$$K_X(t) = \ln\left[\sum_{r=0}^{\infty} \frac{t^r}{r!} \frac{\alpha^r \Gamma_{(r+1)\beta}}{\Gamma_{1\beta}}\right]$$

Hence the first four cumulants are

$$\kappa_1 = \frac{\alpha\Gamma_{2\beta}}{\Gamma_{1\beta}}$$

$$\kappa_2 = \alpha^2\left[\frac{\Gamma_{3\beta}}{\Gamma_{1\beta}} - \frac{\Gamma^2{}_{2\beta}}{\Gamma^2{}_{1\beta}}\right]$$

$$\kappa_3 = \alpha^3\left[\frac{\Gamma_{4\beta}}{\Gamma_{1\beta}} - \frac{3\Gamma_{3\beta}\Gamma_{2\beta}}{\Gamma^2{}_{1\beta}} + \frac{2\Gamma^3{}_{2\beta}}{\Gamma^3{}_{1\beta}}\right]$$

$$\kappa_4 = \alpha^4\left[\frac{\Gamma_{5\beta}}{\Gamma_{1\beta}} - \frac{4\Gamma_{4\beta}\Gamma_{2\beta}}{\Gamma^2{}_{1\beta}} + \frac{6\Gamma_{3\beta}\Gamma^2{}_{2\beta}}{\Gamma^3{}_{1\beta}} - \frac{3\Gamma^4{}_{2\beta}}{\Gamma^4{}_{1\beta}} - 3\left(\frac{\Gamma_{3\beta}}{\Gamma_{1\beta}} - \frac{\Gamma^2{}_{2\beta}}{\Gamma^2{}_{1\beta}}\right)^2\right]$$

Estimation of LBWWD

Newby's methods along with method of moments have been considered to estimate the values of the parameters. Newby [19] took the coefficient of variation which is independent of α, to estimate c by solving, either a graph, or a table or by applying the Newton Raphson algorithm. Here table have been used to estimate c.

It follows that the modified moment equations for the weighted Weibull distribution of order β are

$$\text{cov}^{**} = \text{cov} \qquad \text{(a)}$$

where cov** is the coefficient of variation of length-biased weighted Weibull distribution and cov, coefficient of variation of Weibull distribution. Hence

$$\hat{\alpha} = \frac{\bar{x}\Gamma\left(\beta+\frac{1}{c}+1\right)}{\Gamma\left(\beta+\frac{2}{c}+1\right)} \qquad (b)$$

where equation (a) is solved by using table table for the shape parameter c, which is then substituted into equation (b) providing the scale parameter α. Equation (b) derives from equating the sample mean ($\bar{x}$) to the first raw moment μ_1'.

Application of LBWWD

To justify the suitability of (5) in a practical application, 30 consecutive years data from 1981 - 2010 of June rainfall (in mm) of Tezpur, Assam, India is considered for the study.

For the problem chosen, N = 30, $\bar{x}$ = 278 and cov^{**} = 0.285988. The parameter β is of order 2, c is solved using Newby's method, which is tabulated in Table (3), value of α is calculated from (b). Estimated value of α is $\hat{\alpha}$ = 147.7456. From Table (3) estimated value of c is 1.89. Expected frequencies for each age group were computed with the above estimates substituted into (5). These results are displayed in Table (2).

As seen from Table (2), tabulations of expected frequencies provide an excellent fit of the observed data.

Table 2: Rainfall (mm) in Tezpur of June for 30 years

Rainfall (mm)	Observed Frequency	Expected Frequency
<110	1	0.900
110-180	3	2.778
180-250	6	6.819
250-320	8	7.959
>320	12	11.544
Total	30	30
χ^2		0.145442

Table 3: Coefficient of variation of LBWWD

c	cov**	c	cov**
0.50	0.930949336	1.30	0.397440519
0.60	0.786586676	1.40	0.372422994
0.70	0.684668197	1.50	0.350502158
0.80	0.608202058	1.60	0.331118781
0.90	0.548343928	1.70	0.313842219
1.00	0.500000000	1.80	0.298338545
1.10	0.460009728	1.89	0.285680404
1.20	0.426299365	1.90	0.284342417

CONCLUSION

Length-biased weighted Weibull distribution has been studied. At first, the pdf of the WWD have been obtained considering weight as $w(x) = x^{c\beta}$ by the idea proposed by Rao [24]. From the pdf of WWD, LBWWD have been obtained considering weight as w(x) = x. Next, to characterize the distribution of a random variable *X* of the LBWWD three functions have been introduced; namely the survival function, the probability density or mass function and the failure rate function or hazard function. The moments, the coefficient of variation, the coefficient of skewness and the coefficient of Kurtosis of the LBWWD have been derived. For estimating the parameters of the LBWWD, Newby's [19] method along with method of moments have been used. The LBWWD have been fitted to 30 consecutive years data from 1981-2010 of June rainfall (in *mm*) of Tezpur, Assam, India. LBWWD suggested a good fit of the data. In the environmental and atmospheric sciences LBWWD is found to be suitable use for the practitioner.

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