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Advances in Applied Science Research, 2011, 2 (4):383-390



### A unified study of fractional kinetic equation and thermonuclear functions

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#### ABSTRACT

*The paper discusses the solution of a simple kinetic equation of the type used for the computation of the change of the chemical composition in stars like the sun. Starting from the standard form of the kinetic equation it is generalized to a fractional kinetic equation and its solutions in terms of  $\overline{H}$ -functions are obtained. The role of thermonuclear functions, which are also represented in terms of  $H$ - and  $\overline{H}$ -functions. In such a fractional kinetic equation is emphasized. Results contained in this paper are related to recent investigations of possible astrophysical solutions of the solar neutrino problem.*

**Key words:** Fractional kinetic equation, Thermonuclear function,  $\overline{H}$ -function. (2000 Mathematics subject classification: 33C99).

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#### INTRODUCTION

A spherically symmetric, non-rotating, non-magnetic, self-gravitating model of a star like the sun is assumed to be in thermal equilibrium and hydrostatic equilibrium, with a non-uniform chemical composition throughout. The star is characterized by its mass, luminosity, effective surface temperature, radius, central density, and central temperature. For a given mass, four of these variables are independent and are governed by four simultaneous, non-linear, ordinary differential equations of the first order and four boundary conditions. Since there are four equations but more than four unknowns, additional information must be provided through the equation of state, nuclear energy generation rate, and the opacity (constitutive equations). The assumptions of thermal equilibrium and hydrostatic equilibrium imply that there is no time dependence in the equations describing the internal structure of the star (Kourganoff, 1973, Perdang, 1976, Clayton, 1983).

The evolution of a star like the Sun is governed by a second system of differential equations, the kinetic equations, describing the rate of change of chemical composition of the star for each species in terms of the reaction rates for destruction and production of that species (Kourganoff, 1973, Perdang, 1976, Clayton, 1983).

Methods for modeling processes of destruction and production have been developed for biochemical reactions and their unstable equilibrium states (Murray, 1989) and for chemical reaction networks with unstable states, oscillations, and hysteresis (Nicolis and Prigogine, 1977). Stability investigations of thermonuclear reactions of stellar interest have not yet been worked out in details. However, the potentiality of instabilities in thermonuclear chains may not be overlooked, since, as was pointed out once by Eddington, "what is possible in the (Cavendish) Laboratory may not be too difficult in the Sun" (Perdang, 1976, Mestel, 1999).

Consider an arbitrary reaction characterized by a time dependent quantity  $N = N(t)$ . It is possible to equate the rate of change  $dN/dt$  to a balance between the destruction rate  $d$  and the production rate  $p$  of  $N$ , that is  $dN/dt = -d + p$ . In general, through feedback or other interaction mechanisms, destruction and production depend on the quantity  $N$  itself:  $d = d(N)$  or  $p = p(N)$ . This dependence is complicated since the destruction or production at time  $t$  depends not only on  $N(t)$  but also on the past history  $N(\tau)$ ,  $\tau < t$ , of the variable  $N$ . This may be formally represented by

$$\frac{dN}{dt} = -d(N_t) + p(N_t), \quad (1)$$

Where  $N_t$  denotes the function defined by  $N_t(t^*) = N(t - t^*)$ ,  $t^* > 0$ . Here  $d$  and  $p$  are functional and eq. (1) represents a functional-differential equation. In the following we study a special case of this equation, namely the equation

$$\frac{dN}{dt} = -\alpha N(t) \quad (2)$$

With a constant  $\alpha > 0$ . Eq. (2) implies that spatial fluctuations or in homogeneities in the quantity  $N(t)$  are neglected. The standard solution of the differential equation (2) will be briefly discussed in Section 2 and the generalization to a fractional differential equation and its solution will be derived in Section 3.

## 2. Standard Kinetic Equation

The production and destruction of species is described by kinetic equations governing the change of the number density  $N_i$  of species  $i$  over time, that is,

$$\frac{dN_i}{dt} = -\sum_j N_i N_j \langle \sigma v \rangle_{ij} + \sum_{k,l \neq i} N_k N_l \langle \sigma v \rangle_{kl}, \quad (3)$$

Where  $\langle \sigma v \rangle_{mn}$  denotes the reaction probability for an interaction involving species  $m$  and  $n$ , and the summation is taken over all reactions which either produce or destroy the species

$i$  (Haubold and Mathai, 1995). For a gas of mass density  $\rho$ , the number density  $N_i$  of the species  $i$  is expressed in terms of its abundance  $X_i$ , by the relation  $N_i = \rho N_A X_i / A_i$ , where  $N_A$  is Avogadro's number and  $A_i$  is the mass of  $i$  in mass units. The mean lifetime  $\tau_j(i)$  of species  $i$  for destruction by species  $j$  is given by the relation

$$\lambda_j(i) = \frac{1}{\tau_j(i)} = N_j \langle \sigma v \rangle_{ij} = \rho N_A \frac{X_j}{A_j} \langle \sigma v \rangle_{ij} \tag{4}$$

Where  $\lambda_j(i)$  is the decay rate of  $i$  for interactions with  $j$ . Eq. (4) reveals the Physical importance of  $\langle \sigma v \rangle_{ij}$  for the kinetic equation (3).

In the case of a non-degenerate, non-relativistic gas, if a non-resonant charged nuclear reaction proceeds at low energies dominated by Coulomb-barrier penetration, the reaction probability  $\langle \sigma v \rangle_{mn}$  takes the form (Clayton, 1983, Bergstroem et al., 1999)

$$\langle \sigma v \rangle_{mn} = \left( \frac{8}{\pi \mu} \right)^{1/2} \sum_{\nu=0}^2 \frac{1}{(kT)^{-\nu+1/2}} \frac{S^{(\nu)}(0)}{\nu!} I_2(\nu, a, z, \rho) \tag{5}$$

Where  $I_2$  represents a thermonuclear function given by

$$I_2(\nu-1, a, z, \rho) = \int_0^\infty y^{\nu-1} e^{-ay - zy^{-\rho}} dy, \quad \nu, z, \rho > 0$$

$$= \frac{a^{-\nu}}{\rho} \overline{H}_{0,2}^{2,0} \left[ az \frac{1}{\rho} \middle| \begin{matrix} - \\ (\nu, 1), (0, \frac{1}{\rho}; 1) \end{matrix} \right], \tag{6}$$

Where  $\overline{H}$ -function is defined in terms of a Mellin-Barnes type integral as

$$\overline{H}_{P,Q}^{M,N} [z] = \overline{H}_{P,Q}^{M,N} \left[ z \middle| \begin{matrix} (a_j; \alpha_j; A_j)_{1,N}, (a_j; \alpha_j)_{N+1,P} \\ (b_j; \beta_j)_{1,M}, (b_j; \beta_j; B_j)_{M+1,Q} \end{matrix} \right]$$

$$= \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \overline{\phi}(\xi) z^\xi d\xi \tag{7}$$

where

$$\overline{\phi}(\xi) = \frac{\prod_{j=1}^M \Gamma(b_j - \beta_j \xi) \prod_{j=1}^N \{\Gamma(1 - a_j + \alpha_j \xi)\}^{A_j}}{\prod_{j=M+1}^Q \{\Gamma(1 - b_j + \beta_j \xi)\}^{B_j} \prod_{j=N+1}^P \Gamma(a_j - \alpha_j \xi)} \tag{8}$$

For further details of  $\overline{H}$ -function, we refer to the original paper of Buschman and Srivastava [2].

When  $\rho$  in eq. (6) is real and rational,  $A_j = B_j = 1$ , then the  $\overline{H}$ -function can be reduced to a G-function by using the multiplication formula for gamma functions.

$$\Gamma(mz) = (2\pi)^{\frac{1-m}{2}} m^{mz - \frac{1}{2}} \prod_{j=0}^{m-1} \Gamma\left(z + \frac{j}{m}\right), m = 1, 2, \dots \tag{9}$$

In the case of Coulomb-barrier penetration (Gamow factor),  $\rho = \frac{1}{2}$ ,  $I_2$  reduces to

$$I_2\left(\nu, a, z, \frac{1}{2}\right) = \frac{a^{-(\nu+1)}}{\pi^{1/2}} G_{0,3}^{3,0} \left[ \frac{az^2}{4} \middle| \begin{matrix} - \\ (\nu+1), a, \frac{1}{2} \end{matrix} \right] \tag{10}$$

Where G denotes Meijer's G-function, which was introduced into mathematical analysis in attempts to give meaning to the generalized hypergeometric function  ${}_pF_q$  in the case  $p > q + 1$ .

Proceeding with eq. (3), the first sum in eq. (3) can also be written as

$$-\sum_j N_i N_j \langle \sigma v \rangle_{ij} = -N_i \left( \sum_j N_j \langle \sigma v \rangle_{ij} \right) = N_i a_i, \tag{11}$$

Where  $a_i$  is the statistically expected number of reactions per unit volume per unit time destroying the species  $i$ . It is also a measure of the speed in which the reaction proceeds. In the following we are assuming that there are  $N_j$  ( $j = 1, \dots, i, \dots$ ) species  $j$  per unit volume and that for a fixed  $N_i$  the number of other reacting species that interact with the  $i$ -th species is constant in a unit volume. Following the same argument we have for the second sum in eq. (3) accordingly,

$$+\sum_{k,l \neq i} N_k N_l \langle \sigma v \rangle_{kl} = +N_i b_i \tag{12}$$

Where  $N_i b_i$  is the statistically expected number of the  $i$ -th species produced per unit volume per unit time for a fixed  $N_i$ . Note that the number density of species  $i$ ,  $N_i = N_i(t)$ , is a function of time while the  $\langle \sigma v \rangle_{mn}$ , containing the thermonuclear functions (see eqs. (5) and (6)), are assumed to depend only on the temperature of the gas but not on the time  $t$  and number densities  $N_i$ . Then eq. (1) implies that

$$\frac{dN_i(t)}{dt} = -(a_i - b_i) N_i(t) \tag{13}$$

For eq. (13) we have three distinct cases,  $c_i = a_i - b_i > 0$ ,  $c_i < 0$ , and  $c_i = 0$  of which the last case says that  $N_i$  does not vary over time, which means that the forward and reverse reactions involving species  $i$  are in equilibrium; such a value for  $N_i$  is called a fixed point and

corresponds to a steady-state behavior. The first two cases exhibit that either the destruction ( $c_i > 0$ ) of species  $i$  or production ( $c_i < 0$ ) of species  $i$  dominates.

For the case  $c_i > 0$  we have

$$\frac{dN_i(t)}{dt} = -c_i N_i(t) \tag{14}$$

With the initial condition that  $N_i(t = 0) = N_0$  is the number density of species  $i$  at time  $t = 0$ , and it follows that

$$N_i(t)dt = N_0 e^{-c_i t} dt . \tag{15}$$

The exponential function in eq. (15) represents the solution of the linear one-dimensional differential equation (14) in which the rate of destruction of exhibit instabilities, oscillations, or chaotic dynamics, in striking contrast to its cousin, the logistic finite-difference equation (Perdance, 1976, Haubold and Mathai, 1995). A thorough discussion of eq. (14) and its standard solution in eq. (15) is given in Kourganoff (1973).

### 3. Fractional Kinetic Equation

In the following, for the sake of brevity, the index  $i$  in eq. (14) will be dropped. The standard kinetic equation (14) can be integrated

$$N(t) - N_0 = -c \int_0^t N(\tau) d\tau, \tag{16}$$

Where  $\int_0^t$  is the standard Riemann integral operator. The generalization of this operator to the fractional integral of operator  $p > 0$  is denoted by  ${}_a D_t^{-p}$  and is defined, following Riemann-Liouville, based on the Cauchy formula, by

$${}_a D_t^{-p} f(t) = \frac{1}{\Gamma(p)} \int_a^t f(\tau) (t - \tau)^{p-1} d\tau, p > 0 \tag{17}$$

With

$${}_a D_t^0 f(t) = f(t)$$

(Oldham and Spanier, 1974, Miller and Ross, 1993). The most general fractional integral operator of the type (17) contains  $\overline{H}$ -function (7) as the kernel function. If  $f(t)$  is continuous for  $t \geq a$ , then integration of arbitrary real order has the property

$${}_a D_t^{-p} ({}_a D_t^{-q} f(t)) = {}_a D_t^{-p-q} f(t).$$

Replacing the Riemann integral operator by the fractional Riemann-Liouville operator  ${}_0 D_t^{-\nu}$  in eq. (16), we obtain a fractional integral equation corresponding to eq. (14)

$$N(t) - N_0 = -c^\nu {}_0D_t^{-\nu} N(t). \tag{18}$$

For dimensional reasons, the coefficient  $c$  in eq. (16), containing the probabilities of the reaction under consideration, had to be replaced by  $c^\nu$  accordingly.

The Laplace transform of the Riemann-Liouville fractional integral is

$$L\{ {}_0D_t^{-\nu} f(t); p \} = p^{-\nu} F(p), \tag{19}$$

Where

$$F(p) = \Gamma(\nu) \int_0^\infty e^{-p\tau} f(\tau) d\tau.$$

In order to solve eq. (18), the integral equation is exposed to a Laplace transformation leading to

$$N(p) = L\{N(t); p\} = N_0 \frac{p^{-1}}{1 + \left(\frac{p}{c}\right)^{-\nu}}. \tag{20}$$

To arrive at a representation of eq. (20) in terms of  $\overline{H}$ -function, result can be used,

$$\frac{z^\beta}{1 + az^\alpha} = a^{-\beta/\alpha} \overline{H}_{1,1}^{1,1} \left[ az^\alpha \left| \begin{matrix} (\frac{\beta}{\alpha}, 1; 1) \\ (\frac{\beta}{\alpha}, 1) \end{matrix} \right. \right],$$

$$N(p) = N_0 \frac{1}{c} \overline{H}_{1,1}^{1,1} \left[ \left(\frac{c}{p}\right)^\nu \left| \begin{matrix} (1, 1; 1) \\ (1, 1) \end{matrix} \right. \right] \tag{21}$$

To prepare eq. (21) for an inverse Laplace transform, the following two fundamental properties of an  $\overline{H}$ -function can be used,

$$\frac{1}{k} \overline{H}_{P,Q}^{M,N} \left[ z \left| \begin{matrix} (a_j; \alpha_j; A_j)_{1,N}, (a_j; \alpha_j)_{N+1,P} \\ (b_j; \beta_j)_{1,M}, (b_j; \beta_j; B_j)_{M+1,Q} \end{matrix} \right. \right] = \overline{H}_{P,Q}^{M,N} \left[ z^k \left| \begin{matrix} (a_j; \alpha_j; A_j)_{1,N}, (a_j; \alpha_j)_{N+1,P} \\ (b_j; \beta_j)_{1,M}, (b_j; \beta_j; B_j)_{M+1,Q} \end{matrix} \right. \right], \quad k > 0 \tag{22}$$

$$\overline{H}_{P,Q}^{M,N} \left[ z \left| \begin{matrix} (a_j; \alpha_j; A_j)_{1,N}, (a_j; \alpha_j)_{N+1,P} \\ (b_j; \beta_j)_{1,M}, (b_j; \beta_j; B_j)_{M+1,Q} \end{matrix} \right. \right] = \overline{H}_{P,Q}^{M,N} \left[ \frac{1}{z} \left| \begin{matrix} (1-b_j; \beta_j; A_j)_{1,M}, (1-b_j; \beta_j)_{M+1,Q} \\ (1-a_j; \alpha_j)_{1,N}, (1-a_j; \alpha_j)_{N+1,P} \end{matrix} \right. \right], \quad k > 0 \tag{23}$$

Leading to

$$N(p) = N_0 \frac{1}{c^\nu} \overline{H}_{1,1}^{1,1} \left[ \frac{p}{c} \left| \begin{matrix} (1-\frac{1}{\nu}, \frac{1}{\nu}; 1) \\ (1-\frac{1}{\nu}, \frac{1}{\nu}) \end{matrix} \right. \right], \tag{24}$$

Where the  $\overline{H}$ -function is defined in eq. (7).

The Laplace transforms of  $\overline{H}$ -function (7) is given in terms of another  $\overline{H}$ -function by

$$L\{ \overline{H}(z); p \} = \frac{1}{p} \overline{H}_{Q,P+1}^{N+1,M} \left[ p \left| \begin{matrix} (1-b_j; \beta_j; A_j)_{1,M}, (1-b_j; \beta_j)_{M+1,Q} \\ (1,1), (1-a_j; \alpha_j)_{1,N}, (1-a_j; \alpha_j)_{N+1,P} \end{matrix} \right. \right] \tag{25}$$

For  $0 \leq \mu \leq 1$  in (9),and

$$L\{\overline{H}(z); p\} = \frac{1}{p} \overline{H}^{M, N+1} \left[ \frac{1}{P} \left| \begin{matrix} (0,1), (\alpha_j; A_j)_{1,N}, (\alpha_j; A_j)_{N+1,P} \\ (b_j, \beta_j)_{1,M}, (b_j, \beta_j; B_j)_{M+1,Q} \end{matrix} \right. \right] \tag{26}$$

For  $0 \leq \mu \leq 1$  in (9), respectively.

Further, having  $\overline{H}(p)$ , the inverse Laplace transform of this  $\overline{H}$ -function is given by

$$\overline{H}(z) = L^{-1}\{\overline{H}(p), z\} = \frac{1}{z} \overline{H}^{N, M}_{Q, P+1} \left[ z \left| \begin{matrix} (1-b_j; \beta_j; A_j)_{1,M}, (1-b_j; \beta_j)_{M+1,Q} \\ (1-a_j, \alpha_j)_{1,N}, (1-a_j, \alpha_j; A_j)_{N+1,P}, (1,1;1) \end{matrix} \right. \right] \tag{27}$$

For  $0 \leq \mu \leq 1$  in (9), and

$$\overline{H}(z) = L^{-1}\{\overline{H}(p), z\} = \frac{1}{z} \overline{H}^{M, N}_{P+1, Q} \left[ z \left| \begin{matrix} (\alpha_j, \alpha_j; A_j)_{1,N}, (\alpha_j, \alpha_j)_{N+1,P}, (0,1;1) \\ (b_j, \beta_j)_{1,M}, (b_j, \beta_j; B_j)_{M+1,Q} \end{matrix} \right. \right] \tag{28}$$

For  $\mu \geq 1$  in (9), respectively.

The above four Laplace transforms hold for

$$\max_{1 \leq j \leq n} A_j \operatorname{Re} \left( \frac{1-a_p}{\alpha_p} \right) < \min_{1 \leq j \leq m} B_j \operatorname{Re} \left( \frac{b_q}{\beta_q} \right).$$

Applying an inverse Laplace transform to the  $\overline{H}$ -function in eq. (24) gives

$$N(t) = N_0 \frac{1}{\nu} \overline{H}^{1,1}_{1,2} \left[ ct \left| \begin{matrix} (0, \frac{1}{\nu}; 1) \\ (0, \frac{1}{\nu}), (0,1;1) \end{matrix} \right. \right] \tag{29}$$

Which is the solution of the fractional kinetic equation (18). For the  $\overline{H}$ -function in eq. (7) with (29), the following computable representation can be derived. When the poles of

$\prod_{j=1}^m \Gamma(b_j - \beta_j s)$  are simple, that is,

$$\beta_h (b_j + \lambda) \neq \beta_j (b_h + \nu)$$

For  $j \neq h; j, h = 1, \dots, M; \lambda, \nu = 0, 1, 2, \dots$ . Then one obtains the following expansion for the  $\overline{H}$ -function,

$$\begin{aligned} \overline{H}^{M, N}_{P, Q} &= \sum_{h=1}^M \sum_{\nu=0}^{\infty} \frac{\left\{ \prod_{j=1, j \neq h}^M \Gamma \left( b_j - \beta_j \frac{(b_h + \nu)}{\beta_h} \right) \right\}}{\left\{ \prod_{j=M+1}^Q \Gamma \left( 1 - b_j + \beta_j \frac{(b_h + \nu)}{\beta_h} \right) \right\}} \\ &\quad \frac{\left\{ \prod_{j=1}^N \Gamma \left( 1 - a_j - \alpha_j \frac{(b_h + \nu)}{\beta_h} \right) \right\}}{\left\{ \prod_{j=M+1}^Q \Gamma \left( a_j - \alpha_j \frac{(b_h + \nu)}{\beta_h} \right) \right\}} \frac{(-1)^\nu z^{(b_h + \nu)/\beta_h}}{(\nu)! \beta_h}, \end{aligned} \tag{30}$$

Which exists for all  $z \neq 0$  if  $\mu > 0$  and for  $0 < |z| < \beta^{-1}$  if  $\mu = 0$ , where  $\mu$  and  $\beta$  are given in eqs. (9) and (10). Comparing (29) with eq. (30), one obtains the series expansion

$$N(t) = N_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(\nu k + 1)} (ct)^{\nu k}, \quad (31)$$

For  $\nu = 1$ , the exponential solution of the standard kinetic equation (15) is recovered.

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