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A Review on Dirac – Jordan Transformation Theory

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ABSTRACT

In this review we will try to perform a comprehensive summery of Dirac – Jordan transformation theory. This theory indicates that, any quantum state of a particle will determine by a vector of state in abstract space, D. There are very few authors who have discussed this theory in abstract and pure form. For this reason, the aim of this paper is to illustrate the ability of this theory by different experiments, and to show its capability in different consequences. Then, we will try to show also, the specific distinct between this method and other methods.

Keywords: Dirac – Jordan transformation theory, quantum state, vector of state, abstract space.

INTRODUCTION

Rarely, the authors of quantum mechanics books have discussed Dirac – Jordan transformation theory [1] in abstract and pure form [2]. Mostly, in the topics of mathematical tools, quantum mechanics assimilates in a manner with matrices theory that and its operability and ability differentiates as a pure theory is difficult. The subject of this study is that to show the ability of this theory in different discussions and particular differencies of its solution methods with other theories. Principally, applied mathematics in Dirac – Jordan transformation theory is particular and differs with the mathematics present in theorems and relations of wave and matrices theories [3]. Encountering with wave and matrices theories maybe implies, at least, applied mathematics in these theories gives certain relation between them, but there is not the case of Dirac – Jordan transformation theory [4].

Quantum state of a particle in a given time, in Schrödinger's wave theory, was defined by wave function $\psi(\vec{r})$. Probabilistic interpretation of this wave function requires that its square could be integrated, and this leads to study Hilbert, H space [5 and 6]. By the viewpoint of this theory, a microscopic system may be defined by a wave like equation [7]. Based on elementary suggestions of Schrödinger, these waves may be existed as Maxwell waves. Later, it was clarified that, supposing these waves as real ones is erroneous.

In Dirac – Jordan's theory each quantum state of a particle, may be characterized by a state vector depending to an abstract space D (for respect of Dirac) which was named also, state space of a particle [8]. D space is a sub- space of Hilbert H space. Any element or vector of D space, is named Ket vector or simply, Ket and has show by $|\ \rangle$, and for distinct with other Kets, a symbol is placed inside and was shown as $|\psi\rangle$. D space of the states of a particle is defined by correlating a Ket vector, $|\psi\rangle$, of D to any function which may be integrate its square:

$$\psi(\mathrm{r}) \in \mathrm{H} \leftrightarrow |\psi\rangle \in \mathrm{D} \quad (1)$$

The movement of quantum particle is comparable in some way to rotation of this vector. Selecting basic vectors which are formed in this space, there is a quantum definition. It may be simply going from one definition to another and in fact, Dirac – Jordan's theory leads to perform this, and hence it was named transformation theory.

METHODS AND DEFINITIONS

2.1. Scalar product

To each pair of Kets, $|\psi\rangle$ and $|\varphi\rangle$ could be depended a complex conjugate which is a scalar product and has the following properties:

$$\langle\varphi|\psi\rangle = \int \varphi^*(\mathrm{r})\psi(\mathrm{r})\mathrm{dv} \quad (2)$$

$$\langle\varphi|\psi\rangle = \langle\psi|\varphi\rangle^* \quad (3)$$

$$\langle\varphi|\lambda_1\psi_1 + \lambda_2\psi_2\rangle = \lambda_1\langle\varphi|\psi_1\rangle + \lambda_2\langle\varphi|\psi_2\rangle \quad (4)$$

$$\langle\lambda_1\varphi_1 + \lambda_2\varphi_2|\psi\rangle = \lambda_1^*\langle\varphi_1|\psi\rangle + \lambda_2^*\langle\varphi_2|\psi\rangle \quad (5)$$

It is perfectly obvious that scalar product is linear with respect to ψ function, and is anti- linear with respect to φ function.

2.2. D* space

Any element or vector in D^* space, is named Bra vector or simply, Bra and is shown as $\langle\ |$. Then, $\langle\varphi|\psi\rangle$ symbol shows the position of both state vectors from two different sub-spaces, or determines the position of this two sub-spaces with respect to each other, so:

$$|\psi\rangle \in \mathrm{D} \leftrightarrow \langle\varphi| \in \mathrm{D}^* \quad (6)$$

2.3. There is a Bra related with a Ket

Presence of a scalar product will enable to show that an element of D^*, i.e. a Bra, is related to each Ket, $|\varphi\rangle \in \mathrm{D}$. Bra←Ket relation is anti- linear because:

$$\langle\lambda_1\varphi_1 + \lambda_2\varphi_2|\psi\rangle = \lambda_1^*\langle\varphi_1|\psi\rangle + \lambda_2^*\langle\varphi_2|\psi\rangle = \left(\lambda_1^*\langle\varphi_1| + \lambda_2^*\langle\varphi_2|\right)|\psi\rangle \quad (7)$$

$\lambda_1^*\langle\varphi_1| + \lambda_2^*\langle\varphi_2|$ is the Bra depending to $\lambda_1|\varphi_1\rangle + \lambda_2|\varphi_2\rangle$ Ket, thus, Ket→Bra relation is anti- linear.

If λ is a complex number, and $|\psi\rangle$ is a Ket, then $\lambda|\psi\rangle$ also will be a Ket and so:

$$|\lambda\psi\rangle = \lambda|\psi\rangle \quad (8)$$

Also, $\langle\lambda\psi|$ defined a Bra depending to $|\lambda\psi\rangle$ Ket, because the relation between a Bra and a Ket is anti- linear, and this gives:

$$\langle\lambda\psi| = \lambda^*\langle\psi| \quad (9)$$

2.4. There is not a Ket related with a Bra

However, a Bra is correspond to a Ket, but by selecting examples from Hilbert space it is possible to show that we can find the Bras which aren't correspond to Kets. Wave function of free particle corresponds with a Bra, but it is not correspond with any Ket.

Considering a flat wave which is defined by a momentum p_o in the range of $-\frac{l}{2} \le x \le \frac{l}{2}$ as $\varphi(x)$ function tend to zero outside of this range, so:

$$\varphi(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{ip_o x/\hbar} \qquad -\frac{l}{2} \le x \le \frac{l}{2} \qquad (10)$$

Related Ket with φ(χ) is shown as $|\varphi\rangle$, so that:

$$\varphi(x) \in H \leftrightarrow |\varphi\rangle \in D \qquad (11)$$

Then, it may be written as:

$$\frac{1}{2\pi\hbar}\int_{-\frac{l}{2}}^{\frac{l}{2}} e^{-ip_o x/\hbar} e^{ip_o x/\hbar} dx = \frac{l}{2\pi\hbar} \qquad (12)$$

Which will be divergent with $l \to \infty$, thus in this case, any extreme in Hilbert space is not defined. So that:

$$\lim_{l\to\infty} \varphi(x) \notin H \quad \text{and} \quad \lim_{l\to\infty} |\varphi\rangle \notin D \qquad (13)$$

Now, considering $\langle\varphi|$ for $|\varphi\rangle$ and selecting again Ket from D space, $|\psi\rangle \in D$ and thus we have:

$$\langle\varphi|\psi\rangle = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\frac{l}{2}}^{\frac{l}{2}} e^{-ip_o x/\hbar}\psi(x)dx \qquad (14)$$

Right term in above equation, in limit $l \to \infty$, will be Fourier transformation of $\psi(x)$ for $p = p_o$.

$$\lim_{l\to\infty} \langle\varphi|\psi\rangle = \varphi(p_o) \qquad (15)$$

Hence, when $l \to \infty$, $\langle\varphi|$ tend toward a perfectly determined Bra, $\langle\varphi|_{p=p_o}$, so:

$$\lim_{l\to\infty} \langle\varphi| = \langle\varphi|_{p=p_o} \in D^* \qquad (16)$$

RESULTS AND DISCUSSION

3.1. Representation of inverse operator of momentum in D space

Considering the effort performed for defined time as an operator in quantum mechanics, inverse operator of momentum has an important role. Thus, its representation other than Fourier transformation may be effective. Firstly, evolution operator could be defined as $T(\varepsilon) = e^{ip\varepsilon/\hbar}$, then we have:

$$\int T d\varepsilon = \frac{\hbar}{i}\frac{1}{p} e^{ip\varepsilon/\hbar} = \frac{\hbar}{i}\frac{1}{p}\left[1 + \frac{i\varepsilon}{\hbar}p + o(\varepsilon^2) + \cdots\right] = \frac{\hbar}{i}\frac{1}{p} + \varepsilon + o(\varepsilon^2) + \cdots \qquad (17)$$

This results in:

$$\frac{1}{p} = \frac{i}{\hbar}\left[\int T d\varepsilon - \varepsilon - o(\varepsilon^2)\cdots\right] \qquad (18)$$

Then:

$$\langle x|\frac{1}{p}|\psi\rangle = \frac{i}{\hbar}\left[\langle x|\int T d\varepsilon - \varepsilon|\psi\rangle - \langle x|o(\varepsilon^2)|\psi\rangle \cdots\right] \qquad (19)$$

With $\varepsilon \to 0$ and $d\varepsilon \to dx$ we have:

$$\langle x|\frac{1}{p}|\psi\rangle = \frac{i}{\hbar}\langle x|\int dx|\psi\rangle = \frac{i}{\hbar}\int dx\langle x|\psi\rangle = \frac{i}{\hbar}\int \psi dx \qquad (20)$$

Therefore:

$$\langle x|\frac{1}{p}|\psi\rangle = \left(\frac{i}{\hbar}\int dx\right)\psi \qquad (21)$$

and,

$$\frac{1}{p} = \frac{i}{\hbar}\int dx \qquad (22)$$

3.2. Dimensions of quantum space

Assuming that $\hat{x}$ operator is an eigenstate non zero $|x'\rangle$ with *x'* eigenvalue, so:

$$x|x'\rangle = x'|x'\rangle \qquad (23)$$

The spectrum of $\hat{x}$ operator may be studied. Considering property of evolution operator we have:

$$T(\lambda)|x'\rangle = |x'+\lambda\rangle \quad , \quad [x, T(\lambda)] = \lambda T \qquad (24)$$

Where *T* is evolution operator of transformation, and λ is a true constant. Then:

$$xT|x'\rangle = (x'+\lambda)T|x'\rangle \qquad (25)$$

$T|x'\rangle$ is an eigenstate non zero, and $x'+\lambda$ is an eigenvalue for *x*. Thus, by starting from an eigenstate for *x*, by applying $T(\lambda)$, by any true value, it may be establishing another eigenstate for *x* (in fact, λ may have any true value). Therefore, *x* spectrum is a continuous one which is formed of all possible values over true axis. This shows that in a D space with limited dimension *N*, any *x* and *p* observables are not present which their replacer is $i\hbar$.

Because, any time, by applying transformation operator, without any limitation λ will be accrued. Evidently, eigenvalue for x could not be simultaneously smaller than or equal to space dimension and in same time is infinity. So, space dimension, necessarily, must be infinity.

3.3. Two distinct Kets in D space

Comparing two Kets $|-x'\rangle$ and $-|x'\rangle$ in D space, is useful. Considering the definition of wave function in transformation theory, we can write:

$$\langle x' - |\psi\rangle = \psi(-x') \qquad (26)$$

$$(\langle x'|-)|\psi\rangle = -\langle x'|\psi\rangle = -\psi(x') \qquad (27)$$

As we have:

$$\langle x'|(-|x\rangle) = -\delta(x-x') \text{ and } \langle x|-x'\rangle = \delta(x+x') \qquad (28)$$

Then can write:

$$\psi(x') = \delta(x - x') \text{ and } \psi(-x') = \delta(x + x') \quad (29)$$

So that, the wave functions corresponded to these both Kets are completely different.
Now we define figure operator such as $\phi_x = |x\rangle\langle x|$ and affect it on these both Kets (assuming that $|x'\rangle$ is eigenstate of *x*):

$$\phi_x |-x'\rangle = |x\rangle\langle x|-x'\rangle = k|x\rangle \quad (30)$$

$$\phi_x (-|x'\rangle) = |x\rangle\langle x|(-|x'\rangle) = |x\rangle\langle x|x'\rangle = -k|x\rangle \quad (31)$$

Where, *k* is a constant. It is evident that $|-x'\rangle$ vector is in the same direction with basis Ket $|x\rangle$ and $-|x'\rangle$ vector is in the opposite direction.

3.4. Two mathematical methods in D space

1. According to definition of wave function in this theory $\langle x'|\psi\rangle = \psi(x')$ we have:

$$\left\langle \phi \frac{d}{dx} \middle| x' \right\rangle = \phi(x') \frac{d}{dx'} \quad (32)$$

$$\int \left\langle \phi \frac{d}{dx} \middle| x' \right\rangle \psi(x') dx' = \int \phi(x') \frac{d}{dx'} \psi(x') dx' \quad (33)$$

Above integral may be solved by part to part integrating, as following:

$$u = \phi(x') \qquad du = \frac{d}{dx'} \phi(x') dx' \quad (34)$$

$$v = \psi(x') \qquad dv = \frac{d}{dx'} \psi(x') dx' \quad (35)$$

This gives:

$$[\phi(x')\psi(x')]_{-\infty}^{+\infty} - \int \psi(x') \frac{d}{dx'} \phi(x') dx' \quad (36)$$

Therefore:

$$\int \phi(x') \frac{d}{dx'} \psi(x') dx' = -\int \psi(x') \frac{d}{dx'} \phi(x') dx' \quad (37)$$

$$\phi(x') \frac{d}{dx'} = -\frac{d}{dx'} \phi(x') \quad (38)$$

$$\left\langle \phi \frac{d}{dx} \middle| x' \right\rangle = -\left\langle \frac{d}{dx} \phi \middle| x' \right\rangle \quad (39)$$

Assuming that $|x'\rangle$ is a basic Ket, then we have:

$$\left\langle \phi \frac{d}{dx} \right| = -\left\langle \frac{d}{dx}\phi \right| \qquad (40)$$

2. Supposing $|a'\rangle$ is eigenstate of $\hat{A}$ ($\hat{A}$ is an arbitrary operator), and this gives:

$$\hat{A}|a'\rangle = a'|a'\rangle \qquad (41)$$

And:

$$\langle a'|\hat{A} = \langle a'|a' \qquad (42)$$

Multiplying relation (40) by $\langle a'|$, we obtain:

$$\hat{A}|a'\rangle\langle a'| = a'|a'\rangle\langle a'| \qquad (43)$$

Thus, we can conclude:

$$\left[\hat{A}, |a'\rangle\langle a'|\right] = 0 \qquad (44)$$

And this gives:

$$\left[f(A), |a'\rangle\langle a'|\right] = 0 \qquad (45)$$

Considering $|a'\rangle\langle a'| = \phi_{a'}$, then:

$$f(A)\phi_{a'} = \phi_{a'} f(A) = \phi_{a'} f(A)\phi_{a'} \qquad (46)$$

This equality plays an important role in perturbation theory in quantum mechanics.

CONCLUSION

In this paper the review of Dirac- Jordan transformation theory by different methods, and we have performed a comprehensive summery of this theory. We approach to show that any quantum state of a particle will determine by a vector of state in abstract D space.

This study shows that only by change in wave and matrix theory symbolizing, we obtain a new theorem which consists of a new expression (language) to expose quantum mechanics. Because not only the pervious concepts may be discussed, but, demonstration the concepts as space dimension and differentiation between states, may be carry out stronger.

Dirac- Jordan's theory, at our point of view, is a strong expression, and this review may be a starting point for profound studies generalizing this theory.

REFERENCES

[1] Dirac, P.A.M. (January 1927)., *Proceedings of the Royal Society of London*, (**2011**), A 113 (765), pp. 621.
[2] P.A.M. Dirac, *The Principle of Quantum Mechanics*, Clarendon Press, Oxford, (**1981**).
[3] Duncan A., Janssen M., *Preprint submitted to Elsevier Science*, (1 November **2009**).
[4]. Oppenheimer J. R., *Physical review*, (**1928**), Vol. 13, pp. 66.
[5] Nicholas Young, *An introduction to Hilbert space*, Cambridge University Press, (**1988**).
[6] Joachim Weidmann, *Linear operators in Hilbert spaces, Graduate Texts in Mathematics*, 68, Berlin, New York: Springer-Verlag, (**1980**).
[7] Eduard Prugovečki Eduard, (1981), *Quantum mechanics in Hilbert space* (2nd ed.), Dover Books on Physics, (published **2006**).

[8 J. J. Sakurai, *Modern Quantum Mechanics*, edited by San Fu Tuan, Benjamin/Cummings, Menlo Park, CA, (**1985**).