A numerical approach of a typical three species syn-eco-system

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ABSTRACT

In this paper, we are studying the numerical approach of a typical three species syn-eco-system. The system comprises of a commensal (S1), two hosts S2 and S3 i.e., S2 and S3 both benefit S1, without getting themselves effected either positively or adversely. Further S2 is a commensal of S3 and S3 is a host of both S1, S2. Limited resources are considered for all three species in this case. The model equations of the system constitute a set of three first order non-linear ordinary differential coupled equations. In all, eight equilibrium points of the model are identified. In this paper the numerical solutions for the growth rate equations are computed using Runge-Kutta fourth order method.

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Keywords: Commensal, Equilibrium Point, Host, Runge-Kutta method.

INTRODUCTION

Ecology is the study of the inter-relationships between organisms and environment. It is natural that two or more species living in a common habitat interact in different ways. Mathematical modeling has been playing an important role for the last half a century in explaining several phenomena concerned with individuals and groups of populations in nature. Lotka[14] and Volterra [22] pioneered theoretical ecology significantly and opened new eras in the filed of life and biological sciences. The Ecological interactions can be broadly classified as Ammensalism, Competition, Commensalism, Neutralism, Mutualism, Predation and Parasitism. The general concept of modeling has been presented in the treatises of Meyer[15], Kushing[12], Kapur[10,11]. Srinivas[21] studied competitive ecosystem of two species and three species with limited and unlimited resources. Laxminarayan and Pattabhi Ramacharyulu[13] studied prey-predator ecological models with partial cover for the prey and alternate food for the predator. Archana Reddy[1] and Bhaskara Rama Sharma[2] investigated diverse problems related to two species competitive systems with time delay, employing analytical and numerical techniques. Phani Kumar[16] studied some mathematical models of ecological commensalism. Ravindra Reddy [17] discussed on the stability of two mutually interacting species with mortality rate for the second species. Further, Shiva Reddy et al [20] and Srilatha et al[18,19] studied stability analysis of three and four species. The present authors Hari Prasad and Pattabhi Ramacharyulu [3 to 9] discussed on the stability of a three and four species syn-ecosystems.

The present investigation is on numerical approach of a typical three species (S1, S2, S3) syn-eco system. The system comprises of a commensal (S1), two hosts S2 and S3 i.e., S2 and S3 both benefit S1, without getting themselves effected either positively or adversely. Further S2 is a commensal of S3 and S3 is a host of both S1, S2 where all the three species with limited resources. Figure 1 shows a schematic diagram of the interaction under study.
2. Basic Equations of the Model:
The model equations for a typical three species ecosystem is given by the following system of first order non-linear ordinary differential equations employing the following notation.

**Notation Adopted**

- $S_1$: Commensal of $S_2$ and $S_3$
- $S_2$: Host of $S_1$ and commensal of $S_3$
- $S_3$: Host of $S_1$ and $S_2$
- $N_i(t)$: The population strength of $S_i$ at time $t$, $i = 1, 2, 3$.
- $t$: Time instant.
- $a_i$: Natural growth rate of $S_i$, $i = 1, 2, 3$.
- $a_{ii}$: Self inhibition coefficients of $S_i$, $i = 1, 2, 3$.
- $a_{12}, a_{13}$: Interaction coefficients of $S_1$ due to $S_2$ and $S_1$ due to $S_3$.
- $a_{23}$: Interaction coefficient of $S_2$ due to $S_3$.
- $k_i = \frac{a_i}{a_{ii}}$: Carrying capacities of $S_i$, $i = 1, 2, 3$.
- $t^*$: The dominance reversal time.

Further the variables $N_1, N_2, N_3$ are non-negative and the model parameters $a_1, a_2, a_3, a_{11}, a_{22}, a_{33}, a_{23}, a_{23}$ are assumed to be non-negative constants.

The model equations for the growth rates of $S_1, S_2, S_3$ are

\[
\frac{dN_1}{dt} = a_1 N_1 - a_{11} N_1^2 + a_{12} N_1 N_2 + a_{13} N_1 N_3 \quad (2.1)
\]

\[
\frac{dN_2}{dt} = a_2 N_2 - a_{22} N_2^2 + a_{23} N_2 N_3 \quad (2.2)
\]

\[
\frac{dN_3}{dt} = a_3 N_3 - a_{33} N_3^2 \quad (2.3)
\]

3. Equilibrium States:
The system under investigation has 8 equilibrium states given by

\[
\frac{dN_i}{dt} = 0, \quad i = 1, 2, 3 \quad (3.1)
\]

(i) Fully washed out state

$E_1 : \overline{N}_1 = 0, \overline{N}_2 = 0, \overline{N}_3 = 0$

(ii) States in which two of the tree species are washed out and third is not.
$E_2 : \bar{N}_1 = 0, \bar{N}_2 = 0, \bar{N}_3 = k_3$

$E_3 : \bar{N}_1 = 0, \bar{N}_2 = k_1, \bar{N}_3 = 0$

$E_4 : \bar{N}_1 = k_1, \bar{N}_2 = 0, \bar{N}_3 = 0$

(iii) Only one of the three species is washed out while the other two are not.

$E_5 : \bar{N}_1 = 0, \bar{N}_2 = k_2 + \frac{a_{23}k_3}{a_{22}}, \bar{N}_3 = k_3$

$E_6 : \bar{N}_1 = k_1 + \frac{a_{13}k_3}{a_{11}}, \bar{N}_2 = 0, \bar{N}_3 = k_3$

$E_7 : \bar{N}_1 = k_1 + \frac{a_{13}k_3}{a_{11}}, \bar{N}_2 = k_2, \bar{N}_3 = 0$

(iv) The co-existent state or normal steady state.

$E_8 : \bar{N}_1 = k_1 + \frac{a_{13}}{a_{11}} \left( k_2 + \frac{a_{23}k_3}{a_{22}} \right) + \frac{a_{13}k_3}{a_{11}}, \bar{N}_2 = k_2 + \frac{a_{23}k_3}{a_{11}}, \bar{N}_3 = k_3$

4. A Numerical solution of the Growth Rate Equations:

The numerical solutions of the growth rate basic equations (2.1), (2.2) and (2.3) have been computed employing the fourth order Runge-Kutta method. Some specific typically chosen values of system parameters characterizing in ecological model under investigation and properly chosen initial conditions. Making use of Mat Lab facility. What follows are the results of numerical computation and these are illustrated and some observations made here under.

Case (i): If $N_{i0} < \frac{K_i}{2}$, $i = 1, 2, 3$.

![Figure 2. Variation of $N_1$, $N_2$, and $N_3$ against time (t) for $a_1=1$, $a_2=2.54$, $a_3=2.46$, $K_1=0.38$, $K_2=1.13$, $K_3=6.65$, $a_{12}=0.46$, $a_{13}=2.15$, $a_{23}=1.63$, $N_{10}=0.1$, $N_{20}=0.5$, $N_{30}=2$.](image)
Figure 3. Variation of $N_1$, $N_2$ and $N_3$ against time ($t$) for $a_1=0.07, a_2=0.13, a_3=0.16, K_1=1.4, K_2=1, K_3=0.8, a_{12}=0.13, a_{13}=0.1, a_{23}=0.83, N_{10}=0.6, N_{10}=0.4, N_{30}=0.2$.

Figure 4. Variation of $N_1$, $N_2$ and $N_3$ against time ($t$) for $a_1=0.088, a_2=0.042, a_3=0.074, K_1=0.119, K_2=0.056, K_3=0.206, a_{12}=0.148, a_{13}=0.049, a_{23}=0.063, N_{10}=0.04, N_{20}=0.02, N_{30}=0.08$. 
Case (ii): If $\frac{K_i}{2} < N_{i0} < K_i$, $i = 1,2,3$.

Figure 5. Variation of $N_1$, $N_2$ and $N_3$ against time (t) for $a_1=1, a_2=2.54, a_3=2.46, K_1=0.38, K_2=1.13, K_3=6.65, a_{12}=0.46, a_{13}=2.15, a_{23}=1.63, N_{10}=0.25, N_{20}=1, N_{30}=3$.

Figure 6. Variation of $N_1$, $N_2$ and $N_3$ against time (t) for $a_1=0.07, a_2=0.13, a_3=0.16, K_1=1.4, K_2=1, K_3=0.8, a_{12}=0.13, a_{13}=0.1, a_{23}=0.83, N_{10}=1, N_{20}=0.8, N_{30}=0.5$.
Figure 7. Variation of $N_1$, $N_2$ and $N_3$ against time ($t$) for $a_1=0.088, a_2=0.042, a_3=0.074, K_1=0.119, K_2=0.056, K_3=0.206, a_{12}=0.148, a_{13}=0.049, a_{23}=0.063, N_{10}=0.08, N_{20}=0.04, N_{30}=0.14$.

Case (iii): If $N_{ij} > K_i$, $i = 1, 2, 3$.

Figure 8. Variation of $N_1$, $N_2$ and $N_3$ against time ($t$) for $a_1=1, a_2=2.54, a_3=2.46, K_1=0.38, K_2=1.13, K_3=6.65, a_{12}=0.46, a_{13}=2.15, a_{23}=1.63, N_{10}=11, N_{20}=9, N_{30}=10$. 
Figure 9. Variation of $N_1$, $N_2$ and $N_3$ against time ($t$) for $a_1=0.07, a_2=0.13, a_3=0.16, K_1=1.4, K_2=1, K_3=0.8, a_{12}=0.13, a_{13}=0.1, a_{23}=0.83, N_{10}=1.8, N_{20}=2.5, N_{30}=8.$

Figure 10. Variation of $N_1$, $N_2$ and $N_3$ against time ($t$) for $a_1=0.088, a_2=0.042, a_3=0.074, K_1=0.119, K_2=0.056, K_3=0.206, a_{12}=0.148, a_{13}=0.049, a_{23}=0.063, N_{10}=3, N_{20}=1.5, N_{30}=2.$
Case (iv): If \( N_{i0} = K_i \), \( i = 1,2,3 \).

![Figure 11](image1.png)

**Figure 11.** Variation of \( N_1 \), \( N_2 \) and \( N_3 \) against time \( t \) for \( a_1=0.05, a_2=0.48, a_3=0.44, K_1=1, K_2=1.6, K_3=2.2, a_{12}=0.09, a_{13}=0.05, a_{23}=0.17, N_{10}=1, N_{20}=1.6, N_{30}=2.2 \).  

![Figure 12](image2.png)

**Figure 12.** Variation of \( N_1 \), \( N_2 \) and \( N_3 \) against time \( t \) for \( a_1=1, a_2=2.54, a_3=2.46, K_1=0.38, K_2=1.13, K_3=6.65, a_{12}=0.46, a_{13}=2.15, a_{23}=1.63, N_{10}=0.38, N_{20}=1.13, N_{30}=6.65 \).
Figure 13. Variation of $N_1$, $N_2$ and $N_3$ against time ($t$) for $a_1=0.07$, $a_2=0.13$, $a_3=0.16$, $K_1=1.4$, $K_2=1$, $K_3=0.8$, $a_{12}=0.13$, $a_{13}=0.1$, $a_{23}=0.83$, $N_{10}=1.4$, $N_{20}=1$, $N_{30}=0.8$.

4.1 Observations of the above graphs:

Situation 1: In this situation the natural birth rates of $S_1$, $S_2$, $S_3$ are increasing order. It is noticed that initially the $S_2$ is dominated $S_1$ up to time instant $t^* = 0.7$ and the $S_3$ up to $t^* = 0.9$ after these dominate times we find reversal of the dominance as shown in Figure 2.

Situation 2: In this situation the first species dominates over the second species up to the time instant $t^* = 3.75$ after which the dominance is reversed. Further the third species with low natural birth rate. (Figure 3).

Situation 3: In this situation all the three species have almost equal birth rates and the second species is dominated by the first which itself dominated by the third. Further we notice that all the three species have low growth rates. (Figure 4).

Situation 4: In this situation initially the $S_1$ dominates by the $S_2$ up to the time $t^* = 0.48$ and the $S_3$ up to the time $t^* = 0.7$ and the dominances are reversed. Further the host($S_3$) of both $S_1$ and $S_2$ is dominated by the commensal($S_1$) after the time $t^* = 0.7$ higher value of the commensal coefficient $a_{13}=2.15$. (Figure 5).

Situation 5: In this situation the host ($S_3$) is always dominated by the commensals $S_1$ and $S_2$. In spite of higher natural growth rate, this may be attributed to lowest carrying capacity of $S_3$ compare with $S_1$. Since the carrying capacity of $S_1$ is greater than that of $S_2$. The $S_1$ dominates over the $S_2$ initially up to the time $t^* = 1$ and also often the time $t^* = 4.2$ in between these two species $S_1$ is dominated $S_2$. This is a case of a weak host the strong commensal. (Figure 6).

Situation 6: In this situation the host ($S_3$) dominates over the $S_1$ and $S_2$, the host ($S_2$) dominated by the commensal($S_1$). Further we notice that all the three species have low growth rates. (Figure 7).

Situation 7: In this situation the host ($S_3$) is dominate over the commensal ($S_2$) and dominated by the common commensal($S_1$). This is a case in which commensal coefficient $a_{13}$ is highest. That is the $S_1$ exact higher advantages from the host ($S_3$) resulting the dominance of $S_1$ over $S_2$. (Figure 8).

Situation 8: This is a situation at the carrying capacity of the host ($S_3$) is lowest. In spite of highest initial value, the common host ($S_3$) monotonically decreases as other two $S_1$ and $S_2$ are benefited by host commensal coefficients $a_{13}$ and $a_{23}$. We notes this steep rise of $S_1$ and steeper rise greater fall common $S_1$ and $S_2$. This may be attributed to higher carrying capacity of $S_1$ and higher host commensal coefficient $a_{12}$ between of $S_1$ and $S_2$, compare to $a_{13}$. (Figure 9).
Situation 9: In this situation the initial value of the common commensal ($S_1$) is highest. Further it is evident that all the three species asymptotically converge to the equilibrium point. (Figure 10).

Situation 10: This is a situation at the carrying capacity of the host ($S_3$) is highest. Initially the second species dominates over the first species up to the time instant $t^* = 3.5$ after which the dominance is reversed. Also the third species dominates over the second and first till the time instant $t^* = 1.2$ and $t^* = 2.7$ respectively and thereafter the dominance is reversed. Further we notice that the first species has the least natural birth rate. (Figure 11).

Situation 11: In this situation the natural birth rates of $S_2$ and $S_3$ are almost equal. It is noticed that initially the $S_2$ is dominated $S_1$ up to time instant $t^* = 0.25$ and the $S_3$ up to $t^* = 0.38$ after these dominate times we find reversal of the dominance. (Fig 12).

Situation 12: In this situation the first species has the least natural birth rate. Initially the first species is dominant over the second species for a short span and from the instant $t^* = 0.9$ to $t^* = 3.1$ the first species is dominant. Further the third species is a weak competitor with no appreciable growth even from the start. (Figure 13).

CONCLUSION

Investigate some relation-chains between the species such as Prey-Predation, Neutralism, Commensalism, Mutualism, Competition and Ammensalism between three species ($S_1$, $S_2$, $S_3$) with the population relations.

The present paper deals with an investigation on numerical approach of a typical three species syn eco-system. The system comprises of a commensal ($S_1$), two hosts $S_2$ and $S_3$ ie., $S_2$ and $S_3$ both benefit $S_1$, without getting themselves effected either positively or adversely. Further $S_2$ is a commensal of $S_3$ and $S_3$ is a host of both $S_1$, $S_2$. It is observed that, the numerical solutions for the growth rate equations are computed using Runge-Kutta fourth order method in four cases.

(i): The initial values of the three species are less than half the respective their carrying capacities.
(ii): The initial values of the three species are lie between half their respective carrying capacities and its carrying capacities.
(iii): The initial values of the three species are greater than their respective carrying capacities.
(iv): The initial values of the three species are equal their respective carrying capacities.

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REFERENCES