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**Advances in Applied Science Research, 2012, 3 (2):757-764**

Advances in Applied Science Research
Pelagia Research Library
ISSN: 0976-8610
CODEN (USA): AASRFC

# A Model of two mutually interacting Species with Mortality Rate for the Second Species

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**ABSTRACT**

*The present paper concerns with a model of two mutually interacting species with limited resources for first species and unlimited resources for second species with mortality rate. The model is characterized by a coupled system of first order non-linear ordinary differential equations. In all two equilibrium points are identified. If the death rate of the second species is greater than its birth rate, it is found that there are three equilibrium points. The criteria for asymptotic stability have been established for all the equilibrium points. The co-existent state is always stable. The solutions of the linearised basic equations are obtained and their trends are illustrated.*

**Key words:** Equilibrium points, Mutualism, Coexistence state, Stability.

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## INTRODUCTION

Mathematical modeling is an important interdisciplinary activity which involves the study of some aspects of diverse disciplines. Biology, Epidemiodology, Physiology, Ecology, Immunology, Bio-economics, Genetics, Pharmocokinetics are some of those disciplines. This mathematical modeling has raised to the zenith in recent years and spread to all branches of life and drew the attention of every one. Mathematical modeling of ecosystems was initiated by Lotka [8] and by Volterra [14]. The general concept of modeling has been presented in the treatises of Meyer [9], Cushing [2], Paul Colinvaux [10], Freedman [3], Kapur [5, 6]. The ecological interactions can be broadly classified as Prey-Predation, Competition, Mutualism and so on. N.C. Srinivas [13] studied the competitive eco-systems of two species and three species with regard to limited and unlimited resources. Lakshmi Narayan [7] investigated the two species prey-predator models and stability analysis of competitive species was investigated by Archana Reddy [1]. Local stability analysis for a two-species ecological mutualism model has been presented by the present author et al. [11, 12]. Recently, stability analysis of three species was carried out by Siva Reddy [18]. Further Srilatha et al. [15, 16] and Hari Prasad et.al [17] studied stability analysis of four species. The present investigation is devoted to the analytical study of a model of two mutually interacting species with mortality rate for the second species.

Before describing a model, first we make the following assumptions:
$N_1$ is the population of the first species, $N_2$, the population of the second species, $a_1$ is the rate of natural growth of the first species, $a_2$ is the rate of natural growth of the second species, $\alpha_{11}$ is the rate of decrease of the first species due to insufficient food, $\alpha_{12}$ is the rate of increase of the first species due to interaction with the second species, $\alpha_{21}$ is the rate of increase of the second species due to interaction with the first species. Further note that the variables $N_1, N_2$ and the model parameters $a_1, a_2$, $\alpha_{11}, \alpha_{12}, \alpha_{21}$ are non-negative and that the rate of difference between the death and birth rates is identified as the natural growth rate with appropriate sign. The model equations for a two species mutualising are governed by a system of non-linear ordinary differential equations.

**2. Basic Equations**:
The basic equations are given by

$$\frac{dN_1}{dt} = a_1 N_1 - \alpha_{11} N_1^{\ 2} + \alpha_{12} N_1 N_2 \tag{2.1}$$

$$\frac{dN_2}{dt} = -a_2 N_2 + \alpha_{21} N_1 N_2 \tag{2.2}$$

Here we come across three equilibrium states:

**I.** $\overline{N}_1 = 0; \overline{N}_2 = 0$,

the state in which both the species are washed out .

**II.** $\overline{N}_1 = \frac{a_1}{\alpha_{11}}; \overline{N}_2 = 0$.

Here the first species ( $N_1$ ) survives while the second species ( $N_2$ ) is washed out.

**III.** $\overline{N}_1 = \frac{a_2}{\alpha_{21}}; \overline{N}_2 = \frac{a_2\alpha_{11} - a_1\alpha_{21}}{\alpha_{12}\alpha_{21}}$.

In this state both the species co-exist and this can exist only when $a_2\alpha_{11} - a_1\alpha_{21} > 0$.

<u>**Equilibrium state I**</u> **(fully washed out state):**

To discuss the stability of equilibrium state $\overline{N}_1 = 0; \overline{N}_2 = 0$, we consider small perturbations $u_1(t)$ and $u_2(t)$ from the steady state, i.e. we write

$$N_1 = \overline{N}_1 + u_1(t), \tag{2.3}$$

$$N_2 = \overline{N}_2 + u_2(t) \tag{2.4}$$

Substituting (2.3) and (2.4) in (2.1) and (2.2), we get

$$\frac{du_1}{dt} = a_1 u_1 - \alpha_{11} u_1^{\ 2} + \alpha_{12} u_1 u_2$$

$$\frac{du_2}{dt} = -a_2 u_2 + \alpha_{21} u_1 u_2$$

After linearization, we get

$$\frac{du_1}{dt} = a_1 u_1 \tag{2.5}$$

and

$$\frac{du_2}{dt} = -a_2 u_2 \tag{2.6}$$

The characteristic equation is

$$(\lambda - a_1)(\lambda + a_2) = 0$$

One root of this equation is $\lambda_1 = a_1$ which is positive and the other root is $\lambda_2 = -a_2$ which is negative. Hence the equilibrium state is **unstable.**

The solutions of equations (2.5) and (2.6) are

$$u_1 = u_{10}\ e^{a_1 t} \tag{2.7}$$

$$u_2 = u_{20}\ e^{-a_2 t} \tag{2.8}$$

where $u_{10}$ , $u_{20}$ are the initial values of $u_1$ and $u_2$ . The solution curves are illustrated in figures 1 and 2.

**Case 1:** $u_{10} > u_{20}$ i.e. initially the first species dominates the second species.
We notice that the first species is going away from the equilibrium point while the second species approaches asymptotically to the equilibrium point. Hence the state is unstable.

**Case 2:** $u_{10} < u_{20}$ i.e. initially the second species dominates the first species.
In this case the second species out numbers the first species till the time,

$$t = t^* = \frac{ln\ \{u_{20} / u_{10}\}}{(a_1 + a_2)}$$

after that the first species out numbers the second species and grows indefinitely while the second species approaches asymptotically to the equilibrium point. Hence the state is unstable. Further the trajectories in the $(u_1, u_2)$ plane are given by

$$\left[\frac{u_1}{u_{10}}\right]^{a_2} = \left[\frac{u_2}{u_{20}}\right]^{-a_1}$$

<u>**Equilibrium state II**</u> **($N_1$ exists while $N_2$ is washed out):**
We have

$$\overline{N}_1 = \frac{a_1}{\alpha_{11}} ; \overline{N}_2 = 0$$

Substituting (2.3) and (2.4) in (2.1) and (2.2), we get

$$\frac{du_1}{dt} = -a_1 u_1 - \alpha_{11} u_1^{\ 2} + \alpha_{12} u_1 u_2 + \frac{a_1 \alpha_{12} u_2}{\alpha_{11}}$$

$$\frac{du_2}{dt} = -a_2 u_2 + \alpha_{21} u_1 u_2 + \frac{a_1 \alpha_{21} u_2}{\alpha_{11}}$$

After linearization, we get

$$\frac{du_1}{dt} = -a_1 u_1 + \frac{a_1 \alpha_{12} u_2}{\alpha_{11}} \tag{2.9}$$

$$\frac{du_2}{dt} = \left[\frac{a_1 \alpha_{21}}{\alpha_{11}} - a_2\right] u_2 \tag{2.10}$$

The characteristic equation is

$$(\lambda + a_1)\ \{\lambda - [\frac{a_1 \alpha_{21}}{\alpha_{11}} - a_2]\} = 0 \tag{2.11}$$

One root of this equation (2.11) is $\lambda_1 = -a_1$ which is negative.

**Case A:** When $\frac{a_1}{a_2} > \frac{\alpha_{11}}{\alpha_{21}}$,

the other root of equation (2.11) is $\lambda_2 = \frac{a_1\alpha_{21}}{\alpha_{11}} - a_2$ which is positive. Hence the equilibrium state is **unstable.**

The trajectories are given by

$$u_1 = \frac{1}{\gamma_1}\left[u_{20}a_1\alpha_{12}e^{\lambda_2 t} + \{u_{10}\gamma_1 - u_{20}a_1\alpha_{12}\}e^{-a_1 t}\right] \quad (2.12)$$

$$u_2 = u_{20}e^{\lambda_2 t} \quad (2.13)$$

where

$$\gamma_1 = a_1[\alpha_{11} + \alpha_{21}] - a_2\alpha_{11}$$

The solution curves are illustrated in **figures 3&4.**

**CASE 1:** For $u_{10} < u_{20}$, we have

In this case the second species is noted to be going away from the equilibrium point while the first species would become extinct at the instant

$$t_1^* = \frac{1}{(\lambda_2 + a_1)} ln\left[\frac{u_{20}\alpha_{12}a_1 - u_{10}\gamma_1}{u_{20}\alpha_{12}a_1}\right]$$

As such the state is **unstable**.

**CASE 2:** If $u_{10} > u_{20}$, we have

Here the first species out numbers the second species till the time,

$$t = t^* = \frac{1}{\lambda_2 + a_1} ln\left\{\frac{u_{10}\alpha_{11}(\lambda_2 + a_1) - u_{20}a_1\alpha_{12}}{u_{20}[\alpha_{11}(\lambda_2 + a_1) - a_1\alpha_{12}]}\right\}$$

there after the second species out numbers the first species. And also the second species is noted to be going away from the equilibrium point while the first species would become extinct at the instant

$$t_1^* = \frac{1}{(\lambda_2 + a_1)} ln\left[\frac{u_{20}\alpha_{12}a_1 - u_{10}\gamma_1}{u_{20}\alpha_{12}a_1}\right]$$

As such the state is **unstable**.

**Case B:** When $\frac{a_1}{a_2} < \frac{\alpha_{11}}{\alpha_{21}}$

One root of the equation (2.11) is $\lambda_1 = -a_1$ which is negative and the other root is

$\lambda_2 = \frac{a_1\alpha_{21}}{\alpha_{11}} - a_2$ which is negative.

As the roots of the equation (2.11) are both negative, the equilibrium state is **stable.**

The trajectories in this case are the same as in (2.12) and (2.13).

That is

$$u_1 = \frac{1}{\gamma_1}\left[u_{20}a_1\alpha_{12}e^{\lambda_2 t} + \{u_{10}\gamma_1 - u_{20}a_1\alpha_{12}\}e^{-a_1 t}\right]$$

$$u_2 = u_{20}e^{\lambda_2 t}$$

where

$$\gamma_1 = a_1[\alpha_{11} + \alpha_{21}] - a_2\alpha_{11}$$

The solution curves are illustrated in **figures 5&6.**

**CASE 1:** $u_{10} < u_{20}$ i.e. initially the second species dominates the first species.
In this case the second species always out numbers the first species. It is evident that both the species converging asymptotically to the equilibrium point. Hence the state is **stable**.

**CASE 2:** $u_{10} > u_{20}$ i.e. initially the first species dominates the second species.
Here the first species out numbers the second species till the time,

$$t = t^* = \frac{1}{\lambda_2 + a_1} ln \left\{\frac{u_{10}\alpha_{11}(\lambda_2 + a_1) - u_{20}a_1\alpha_{12}}{u_{20}[\alpha_{11}(\lambda_2 + a_1) - a_1\alpha_{12}]}\right\}$$

there after the second species out numbers the first species. As $t \to \infty$ both $u_1$ & $u_2$ approach to the equilibrium point. Hence the state is **stable.** Further the trajectories in the $(u_1, u_2)$ plane are given by

$$(q_1 - 1)u_1 = c\,u_2^{\,q_1} + p_1 u_2$$

where

$$p_1 = \frac{a_1\alpha_{12}}{a_2\alpha_{11} - a_1\alpha_{21}};$$

$$q_1 = \frac{a_1\alpha_{11}}{a_2\alpha_{11} - a_1\alpha_{21}}$$

and $c$ is an arbitrary constant.

The solution curves are illustrated in figure 7.

**<u>Equilibrium state III</u> (coexistence state):**
We have

$$\overline{N}_1 = \frac{a_2}{\alpha_{21}}; \overline{N}_2 = \frac{a_2\alpha_{11} - a_1\alpha_{21}}{\alpha_{12}\alpha_{21}} \text{ wherein } a_2\alpha_{11} - a_1\alpha_{21} > 0$$

Substituting (2.3) and (2.4) in (2.1) and (2.2), we get

$$\frac{du_1}{dt} = -\alpha_{11}u_1^2 + \alpha_{12}u_1u_2 - \alpha_{11}\overline{N}_1 u_1 + \alpha_{12}\overline{N}_1 u_2$$

$$\frac{du_2}{dt} = \alpha_{21}u_1\overline{N}_2 + \alpha_{21}u_1u_2$$

After linearization, we get

$$\frac{du_1}{dt} = -\alpha_{11}\overline{N}_1 u_1 + \alpha_{12}\overline{N}_1 u_2 \qquad (2.14)$$

and

$$\frac{du_2}{dt} = \alpha_{21} u_1 \overline{N}_2 \qquad (2.15)$$

The characteristic equation is

$$\lambda^2 + \alpha_{11} \overline{N}_1 \lambda - \alpha_{12}\alpha_{21} \overline{N}_1 \overline{N}_2 = 0$$

One root of this equation is positive and the other root is negative. Hence the equilibrium state is **unstable**.

The trajectories are given by

$$u_1 = \left[\frac{u_{10}\lambda_1 + u_{20}\alpha_{12}\overline{N}_1}{\lambda_1 - \lambda_2}\right] e^{\lambda_1 t} + \left[\frac{u_{10}\lambda_2 + u_{20}\alpha_{12}\overline{N}_1}{\lambda_2 - \lambda_1}\right] e^{\lambda_2 t}$$

$$u_2 = \left[\frac{u_{20}(\lambda_1 + \alpha_{11}\overline{N}_1) + u_{10}\alpha_{21}\overline{N}_2}{\lambda_1 - \lambda_2}\right] e^{\lambda_1 t} + \left[\frac{u_{20}(\lambda_2 + \alpha_{11}\overline{N}_1) + u_{10}\alpha_{21}\overline{N}_2}{\lambda_2 - \lambda_1}\right] e^{\lambda_2 t}$$

The curves are illustrated in **figures 8&9.**

**Case 1:** $u_{10} > u_{20}$ i.e. initially the first species dominates the second species.
In this case, the first species is noted to be going away from the equilibrium point while the second species approaches asymptotically to the equilibrium point. Hence the state is unstable.

**Case 2:** $u_{10} < u_{20}$ i.e. initially the second species dominates the first species.
In this case the second species out numbers the first species till the time,

$$t = t^* = \frac{1}{\lambda_2 - \lambda_1} \ln\left[\frac{(b_2 - \lambda_1)u_{10} + (a_3 - b_1)u_{20}}{(b_2 - \lambda_2)u_{10} + (a_4 - b_1)u_{20}}\right]$$

where

$$b_1 = \alpha_{12}\overline{N}_1; \quad b_2 = \alpha_{21}\overline{N}_2;$$
$$a_3 = \lambda_1 + \alpha_{11}\overline{N}_1; \quad a_4 = \lambda_2 + \alpha_{11}\overline{N}_1$$

after that the first species out numbers the second species and grows indefinitely while the second species approaches asymptotically to the equilibrium point. Hence the state is unstable. Further the trajectories in the $(u_1, u_2)$ plane are given by

$$[u_2^{(a-1)(v_1 - v_2)}]d = \frac{(u_1 - u_2 v_1)^{av_1}}{(u_1 - v_2 u_2)^{av_2}}$$

where $v_1$ and $v_2$ are roots of the quadratic equation $av^2 + bv + c = 0$ with $a = \alpha_{21}\overline{N}_2$ ; $b = \alpha_{11}\overline{N}_1$ ; $c = -\alpha_{12}\overline{N}_1$ and $d$ is an arbitrary constant.

**Trajectories:**

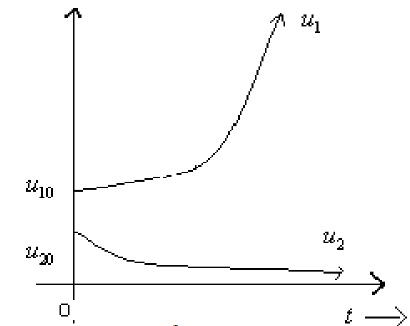

Figure 1

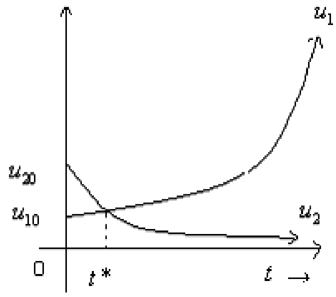

Figure 2

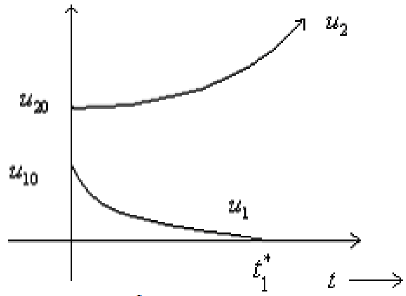

Figure 3

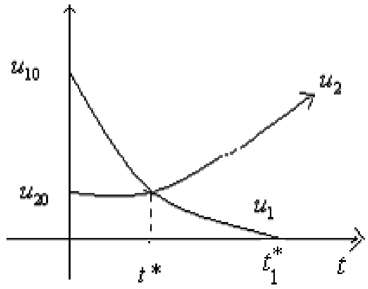

Figure 4

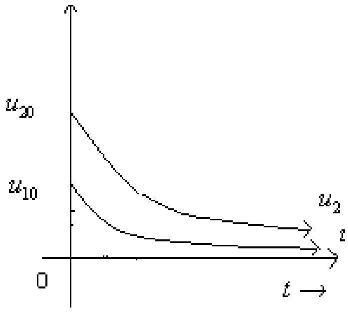

Figure 5

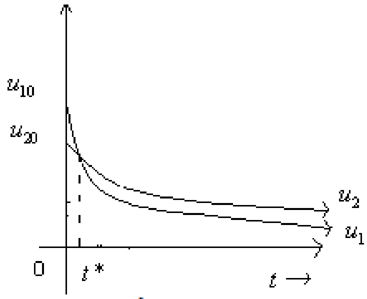

Figure 6

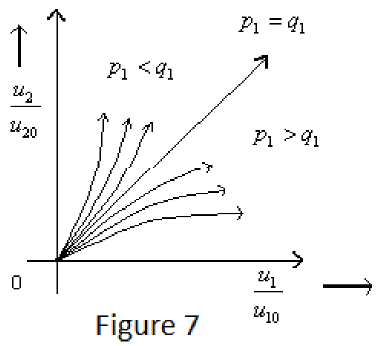

Figure 7

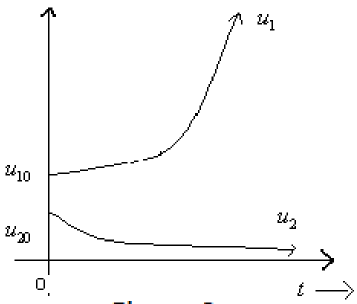

Figure 8

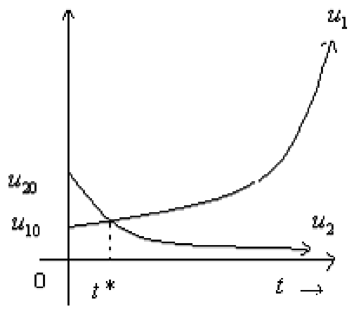

Figure 9

## REFERENCES


[1] Archana Reddy R, On the stability of some mathematical models in biosciences- interacting species, Ph.D thesis, **2009**, JNTU.
[2] Cushing JM, Integro – differential equations and Delay Models in Population Dynamics, Lecture Notes in Biomathematics, Springer–Verlag, Heidelberg, **1977**, Vol. 20.
[3] Freedman HI, Deterministic Mathematical Models in Population Ecology, Marcel – Decker, New York, **1980**.
[4] George F Simmons, Differential Equations with applications and historical notes, Tata McGraw-Hill, New Delhi, **1974**.
[5] Kapur JN, Mathematical Modelling, Wiley – Eastern, **1988**.
[6] Kapur JN, Mathematical Models in Biology and Medicine Affiliated East – West, **1985**.
[7] Lakshmi Narayan K, A Mathematical study of Prey-Predator Ecological Models with a partial covers for the prey and alternative food for the predator, Ph.D thesis, J.N.T.University, **2004**.
[8] Lotka AJ, Elements of Physical biology, Williams and Wilkins, Baltimore, **1925**.
[9] Meyer WJ, Concepts of Mathematical Modelling, Mc Graw – Hill, **1985**.
[10] Paul Colinvaux, Ecology, John Wiley and Sons Inc., New York, **1986**.

[11] Ravindra Reddy B, Lakshmi Narayan K and Pattabhiramacharyulu NCh, *International J.of Engg. Research & Indu. Appls*, Vol.2, No.II (**2009**), 281-291.
[12] Ravindra Reddy B, Lakshmi Narayan K and Pattabhiramacharyulu NCh, *International J. of Math. Sci & Engg. Appls. (IJMSEA),* Vol. 4, No. III (August, **2010**), pp.97-106.
[13] Srinivas NC, Some Mathematical aspects of modeling in Bio Medical Sciences, Ph.D thesis, Kakatiya University, **1991**.
[14] Volterra V, Leconssen la theorie mathematique de la leitte pou lavie, Gauthier – Villars, Paris, **1931**.
[15] Srilatha R, Ravindra Reddy B and Pattabhiramacharyulu NCh, *Advances in Applied Science Research*, **2011**, 2 (3): 151-165.
[16] Srilatha R and Pattabhiramacharyulu NCh, *Advances in Applied Science Research*, **2011**, 2 (3): 166-178.
[17] Hari Prasad B and Pattabhiramacharyulu NCh, *Advances in Applied Science Research*, **2011**, 2 (5): 197-206.
[18] Shiva Reddy K and Pattabhi Ramacharyulu NCh, *Advances in Applied Science Research*, **2011**, 2 (3): 208-218.